1.) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+n} = 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ this is telescoping

w/ $\displaystyle\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \Rightarrow 1 = A(n+1) + Bn$

$n=0 \Rightarrow A = 1$

$n=-1 \Rightarrow B = -1$

so $\displaystyle 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 3\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$

Then $\displaystyle S_n = \sum_{i=1}^{n}\left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$\displaystyle = 1 - \frac{1}{n+1}$

so sum $\displaystyle = 3\lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 3$

2.) $\displaystyle\sum_{n=1}^{\infty}\left(e^{\frac{1}{n}} - e^{\frac{1}{n+1}}\right)$ telescoping

so $\displaystyle S_n = \sum_{i=1}^{n}\left(e^{\frac{1}{i}} - e^{\frac{1}{i+1}}\right) = \left(e - e^{\frac{1}{2}}\right) + \left(e^{\frac{1}{2}} - e^{\frac{1}{3}}\right) + \cdots + \left(e^{\frac{1}{n}} - e^{\frac{1}{n+1}}\right)$

$\displaystyle = e - e^{\frac{1}{n+1}}$

so sum $\displaystyle = \lim_{n\to\infty}\left(e - e^{\frac{1}{n+1}}\right) = e - 1$

3.) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$ use ratio test

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}} \cdot \frac{5^n\sqrt{n}}{(2x-1)^n}\right| = \frac{|2x-1|}{5}\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = \frac{|2x-1|}{5} < 1$ need

$\Rightarrow |x - \frac{1}{2}| < \frac{5}{2} \leftarrow$ Radius so now $-\frac{5}{2} < x - \frac{1}{2} < \frac{5}{2} \Rightarrow -2 < x < 3$

$x = -2$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ conv. by A.S.T.

$x = 3$: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ div. p-series $p = \frac{1}{2}$

Interval $[-2, 3)$ $R = \frac{5}{2}$

4.) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{(x+3)^n}{3^n \ln n}$ use ratio test

via L'Hôpital

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x+3)^{n+1}}{3^{n+1}\ln(n+1)} \cdot \frac{3^n \ln n}{(x+3)^n} \right| = \frac{|x+3|}{3} \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \frac{|x+3|}{3} < 1 \quad \uparrow \text{need}$$

$\Rightarrow$ $|x+3| < 3$ ← Radius so $-3 < x+3 < 3 \Rightarrow -6 < x < 0$

$x = -6$: $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n(-1)^n}{\ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ div. comp to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$

$x = 0$: $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ conv. by A.S.T. Interval: $(-6, 0]$, $R = 3$

5.) $f(x) = \ln(3x)$ $a = 1$ in general $f^{(n)}(1) = (-1)^{n-1}(n-1)!$ for $n \geq 1$

$f'(x) = \frac{1}{x}$ $f(1) = \ln 3$

$f'(1) = 1$ so $c_n = \dfrac{(-1)^{n-1}(n-1)!}{n!} = \dfrac{(-1)^{n-1}}{n}$ for $n \geq 1$

$f''(x) = \frac{-1}{x^2}$ $f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $f'''(1) = 2$ so $f(x) = \ln 3 + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$

$f^{(4)}(x) = \frac{-2\cdot 3}{x^4}$ $f^{(4)}(1) = -2\cdot 3$

6.) $f(x) = \cos(\pi x)$ $a = 1$ so $c_0 = -1$

$f'(x) = -\pi\sin(\pi x)$ $f(1) = -1$ $c_1 = 0$

$f''(x) = -\pi^2\cos(\pi x)$ $f'(1) = 0$ $c_2 = +\pi^2$

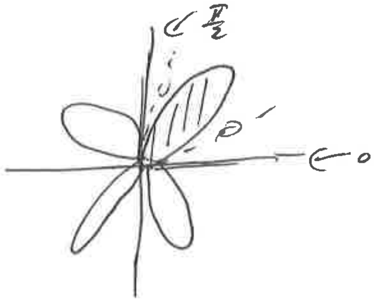
$f'''(x) = \pi^3\sin(\pi x)$ $f''(1) = +\pi^2$ $\vdots$

$f^{(4)}(x) = \pi^4\cos(\pi x)$ $f'''(1) = 0$

$f^{(4)}(1) = -\pi^4$

So $f(x) = -1 + \dfrac{\pi^2}{2!}(x-1)^2 - \dfrac{\pi^4}{4!}(x-1)^4 + \dfrac{\pi^6}{6!}(x-1)^6 + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}\frac{\pi^{2n}}{(2n)!}(x-1)^{2n}$

7.) $r = 4\sin(2\theta)$



$4\sin(2\theta) = 0$

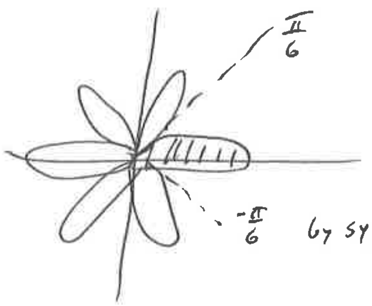
$\Rightarrow \quad 2\theta = 0, \pi, 3\pi, \ldots$

$\theta = 0, \frac{\pi}{2}, \ldots$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} 16\sin^2(2\theta)\,d\theta = \frac{16}{4}\int_0^{\frac{\pi}{2}}(1-\cos(4\theta))\,d\theta$$

$$= 4\left(\theta - \frac{1}{4}\sin(4\theta)\right)\Big|_0^{\frac{\pi}{2}} = 2\pi$$

8.) $r = 6\cos(3\theta)$



by symmetry

$6\cos(3\theta) = 0$

$\Rightarrow \quad 3\theta = \frac{\pi}{2}, \frac{-\pi}{2}, \ldots$

$\theta = \frac{\pi}{6}, \frac{-\pi}{6}, \ldots$

$$A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 36\cos^2(3\theta)\,d\theta = \int_0^{\frac{\pi}{6}} 36\cos^2(3\theta)\,d\theta$$

$$= 18\int_0^{\frac{\pi}{6}}(1+\cos(6\theta))\,d\theta = 18\left(\theta + \frac{1}{6}\sin(6\theta)\right)\Big|_0^{\frac{\pi}{6}}$$

$$= 3\pi$$

9.) $\vec{a} = \langle 3,5,1\rangle, \quad \vec{b} = \langle -1,4,7\rangle, \quad \vec{c} = \langle 5,7,2\rangle$

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})| = \left|\begin{vmatrix} 3 & 5 & 1 \\ -1 & 4 & 7 \\ 5 & 7 & 2 \end{vmatrix}\right| = \left|3\begin{vmatrix} 4 & 7 \\ 7 & 2\end{vmatrix} - 5\begin{vmatrix} -1 & 7 \\ 5 & 2\end{vmatrix} + \begin{vmatrix} -1 & 4 \\ 5 & 7\end{vmatrix}\right|$$

$$= |3(-41) - 5(-37) - 27| = 35$$

10.) $\vec{a} = \langle 1,2,3\rangle, \quad \vec{b} = \langle 1,2,1\rangle, \quad \vec{c} = \langle 1,1,1\rangle$

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})| = \left|\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}\right| = \left|1\begin{vmatrix} 2 & 1 \\ 1 & 1\end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 1 & 1\end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 1 & 1\end{vmatrix}\right|$$

$$= |1 - 0 - 3| = 2$$

11.) $\vec{a}\cdot\vec{b} = \sqrt{2}$, $\vec{a}\times\vec{b} = \langle 1, 0, 1\rangle$

so $\tan\theta = \dfrac{\|\vec{a}\times\vec{b}\|}{\vec{a}\cdot\vec{b}}$ and $\|\vec{a}\times\vec{b}\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$ so $\tan\theta = \frac{\sqrt{2}}{\sqrt{2}} = 1 \Rightarrow \theta = \frac{\pi}{4}$

12.) $\vec{a}\cdot\vec{b} = 10\sqrt{3}$, $\vec{a}\times\vec{b} = \langle\sqrt{20}, 4, 8\rangle$

and $\|\vec{a}\times\vec{b}\| = \sqrt{20+16+64} = \sqrt{100} = 10$ so $\tan\theta = \frac{10}{10\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

13.) $P = (1, 4, 2)$, $Q = (-3, 6, 11)$, $R = (5, 2, -3)$

so $\vec{a} = \vec{PQ} = \langle -4, 2, 9\rangle$,

$\vec{b} = \vec{PR} = \langle 4, -2, -5\rangle$

and $\vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 2 & 9 \\ 4 & -2 & -5 \end{vmatrix} = \langle 8, 16, 0\rangle$

so $\vec{r}_0 = \langle 1, 4, 2\rangle$ and $\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0 \Rightarrow \langle 8, 16, 0\rangle\cdot\langle x-1, y-4, z-2\rangle = 0$

$\Rightarrow 8(x-1) + 16(y-4) = 0 \Rightarrow 8x + 16y - 72 = 0$

14.) $P = (1, 0, 5)$, $Q = (4, -1, 3)$, $R = (9, 7, -2)$

$\vec{a} = \vec{PQ} = \langle 3, -1, -2\rangle$, $\vec{b} = \vec{PR} = \langle 8, 7, -7\rangle$

and $\vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & -2 \\ 8 & 7 & -7 \end{vmatrix} = \langle 21, 5, 29\rangle$

so $\vec{r}_0 = \langle 1, 0, 5\rangle$ and $\langle 21, 5, 29\rangle\cdot\langle x-1, y, z-5\rangle = 0$

$\Rightarrow 21(x-1) + 5y + 29(z-5) = 0 \Rightarrow 21x + 5y + 29z - 166 = 0$

15.) $\vec{r}(t) = \langle 2\sin t, 2\sin(2t), 2\sin(3t)\rangle$ pt $(1, \sqrt{3}, 2)$

$\begin{cases} 1 = 2\sin t \\ \sqrt{3} = 2\sin(2t) \\ 2 = 2\sin(3t) \end{cases} \Rightarrow \begin{matrix} \sin t = \frac{1}{2} & \Rightarrow t = \frac{\pi}{6}, \frac{5\pi}{6}, \dots \\ \sin(2t) = \frac{\sqrt{3}}{2} & t = \frac{\pi}{6}, \frac{2\pi}{3}, \dots \\ \sin(3t) = 1 & t = \frac{\pi}{6}, \frac{\pi}{2}, \dots \end{matrix} \Rightarrow t = \frac{\pi}{6}$

$\vec{r}\,'(t) = \langle 2\cos t, 4\cos(2t), 6\cos(3t)\rangle$

$\vec{v} = \vec{r}\,'(\frac{\pi}{6}) = \langle\sqrt{3}, 2, 0\rangle$ and $\vec{r}_0 = \langle 1, \sqrt{3}, 2\rangle$

so $\vec{r} = \langle 1, \sqrt{3}, 2\rangle + t\langle\sqrt{3}, 2, 0\rangle = \langle 1+t\sqrt{3}, \sqrt{3}+2t, 2\rangle$

16.) $\vec{r}(t) = \langle \sqrt{t^2+3},\ \ln(t^2+3),\ t \rangle$ pt $(2, \ln 4, 1)$

$\begin{cases} 2 = \sqrt{t^2+3} \\ \ln 4 = \ln(t^2+3) \\ 1 = t \end{cases} \Rightarrow t = 1$ and $\vec{r}\,'(t) = \left\langle \frac{t}{\sqrt{t^2+3}},\ \frac{2t}{t^2+3},\ 1 \right\rangle$

so $\vec{v} = \vec{r}\,'(1) = \left\langle \frac{1}{\sqrt{4}},\ \frac{2}{4},\ 1 \right\rangle = \left\langle \frac{1}{2}, \frac{1}{2}, 1 \right\rangle$

and $\vec{r}_0 = \langle 2, \ln 4, 1 \rangle$

so $\vec{r} = \langle 2, \ln 4, 1 \rangle + t\left\langle \frac{1}{2}, \frac{1}{2}, 1 \right\rangle = \left\langle 2 + \frac{t}{2},\ \ln 4 + \frac{t}{2},\ 1 + t \right\rangle$

17.) $\vec{r}(t) = \langle \cos t, \sin t, \ln(\cos t) \rangle$ and $0 \le t \le \frac{\pi}{4}$

so $\vec{r}\,'(t) = \langle -\sin t, \cos t, -\tan t \rangle$ and $\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}$

$= \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t} = \sec t$

so $L = \int_0^{\frac{\pi}{4}} \sec t\, dt = \ln|\sec t + \tan t|\Big|_0^{\frac{\pi}{4}} = \ln(\sqrt{2} + 1)$

18.) $\vec{r}(t) = \langle \sqrt{2}\,t, e^t, e^{-t} \rangle$ $0 \le t \le 1$

$\vec{r}\,'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle$ $\|\vec{r}\,'(t)\| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$

so $L = \int_0^1 (e^t + e^{-t})dt = (e^t - e^{-t})\Big|_0^1 = e - e^{-1}$

19.) $\vec{r}(t) = \langle \cos t, \sin t, 10 \rangle$

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$, $\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$, $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle$, $\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$ so $\vec{T}(t) = \langle -\sin t, \cos t, 0 \rangle$

$\vec{T}\,'(t) = \langle -\cos t, -\sin t, 0 \rangle$, $\|\vec{T}\,'(t)\| = \sqrt{\cos^2 t + \sin^2 t} = 1$ so $\vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle$

and $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \langle 0, 0, 1 \rangle$ so $\vec{B}(t) = \langle 0, 0, 1 \rangle$

20.) $\vec{r}(t) = \langle \cos t, \sin t, \ln(\cos t) \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, -\tan t \rangle$, $\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}$
$= \sqrt{1 + \tan^2 t} = \sec t = \frac{1}{\cos t}$

so $\vec{T}(t) = \langle -\sin t \cos t, \cos^2 t, -\sin t \rangle$

$\vec{T}\,'(t) = \langle -\cos^2 t + \sin^2 t, -2\sin t \cos t, -\cos t \rangle = \langle -\cos(2t), -\sin(2t), -\cos t \rangle$

$\|\vec{T}\,'(t)\| = \sqrt{\cos^2(2t) + \sin^2(2t) + \cos^2 t} = \sqrt{1 + \cos^2 t}$ so $\vec{N}(t) = \frac{1}{\sqrt{1+\cos^2 t}} \langle -\cos(2t), -\sin(2t), -\cos t \rangle$

so $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{\sqrt{1+\cos^2 t}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t \cos t & \cos^2 t & -\sin t \\ -\cos(2t) & -\sin(2t) & -\cos t \end{vmatrix}$

$= \frac{1}{\sqrt{1+\cos^2 t}} \langle -\cos^3 t - \sin t \sin(2t), \sin t \sin(2t) - \sin t \cos^2 t, \sin(2t) \sin t \cos t + \cos^2 t \sin(2t) \rangle$

21.) $\vec{r}(t) = \left\langle \int_0^t \sin\left(\frac{\pi}{2}u^2\right) du, \int_0^t \cos\left(\frac{\pi}{2}u^2\right) du \right\rangle$

$\vec{r}\,'(t) = \langle \sin(\frac{\pi}{2}t^2), \cos(\frac{\pi}{2}t^2) \rangle$ reparametrize w.r.t. arc length

$s = \int_0^t \|\vec{r}\,'(u)\| du$ but $\|\vec{r}\,'(t)\| = \sqrt{\sin^2(\frac{\pi}{2}t^2) + \cos^2(\frac{\pi}{2}t^2)} = 1$

so $s = \int_0^t du = t$ so $t = s$ and $\vec{r}\,'(s) = \langle \sin(\frac{\pi}{2}s^2), \cos(\frac{\pi}{2}s^2) \rangle$

so $\vec{T}\,'(s) = \langle \pi s \cos(\frac{\pi}{2}s^2), -\pi s \sin(\frac{\pi}{2}s^2) \rangle$

and $\vec{T}(s) = \langle \sin(\frac{\pi}{2}s^2), \cos(\frac{\pi}{2}s^2) \rangle$ so

so $\kappa = \|\vec{T}\,'(s)\| = \sqrt{(\pi s)^2 (\cos^2(\frac{\pi}{2}s^2) + \sin^2(\frac{\pi}{2}s^2))} = \pi s$