Practice Exam 1:

1.) $\begin{cases} u_t = k u_{xx} \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$

let $u(x,t) = h(x) g(t)$ $\Rightarrow$ $h(x) \frac{dg}{dt} = k\, g(t) \frac{d^2h}{dx^2}$ $\Rightarrow$ $\frac{1}{k g(t)} \frac{dg}{dt} = \frac{1}{h(x)} \frac{d^2 h}{dx^2} = -\lambda$

$\Rightarrow$ ① $\frac{dg}{dt} = -k\lambda g$ and ② $\begin{cases} \frac{d^2h}{dx^2} + \lambda h = 0 \\ \frac{dh}{dx}(0) = \frac{dh}{dx}(L) = 0 \end{cases}$

so ① $\Rightarrow$ $g(t) = C e^{-k\lambda t}$

② $\frac{d^2h}{dx^2} + \lambda h = 0$ $\Rightarrow$ $r^2 + \lambda = 0$ so $r = \pm\sqrt{-\lambda}$

case: $\lambda = 0$ $\Rightarrow$ $r = 0, 0$ $h(x) = c_1 + c_2 x$, $\frac{dh}{dx} = c_2$ $\frac{dh}{dx}(0) = \frac{dh}{dx}(L) = c_2$ by B.C. $\Rightarrow$ $c_2 = 0$

so $h(x) = c_1$

case: $\lambda < 0$ $\Rightarrow$ let $-\lambda = s^2 > 0$ so $r = \pm s$ $h(x) = c_1 e^{sx} + c_2 e^{-sx}$

$\frac{dh}{dx} = s(c_1 e^{sx} - c_2 e^{-sx})$ $0 = \frac{dh}{dx}(0) = s(c_1 - c_2)$ $s \neq 0$ by assumption so $c_1 = c_2$

$\Rightarrow$ $\frac{dh}{dx} = s c_1 (e^{sx} - e^{-sx})$ so $0 = \frac{dh}{dx}(L) = s c_1 (e^{sL} - e^{-sL})$ $s \neq 0$ assume and want $c_1 \neq 0$

$\Rightarrow$ $e^{sL} = e^{-sL}$ $\Rightarrow$ $s = 0$ contradiction $\Rightarrow$ $c_1 = 0$ so $c_2 = 0$

case: $\lambda > 0$ $\Rightarrow$ $r = \pm i\sqrt{\lambda}$ so $h(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$

$\frac{dh}{dx} = \sqrt{\lambda}\left(c_2 \cos(\sqrt{\lambda} x) - c_1 \sin(\sqrt{\lambda} x)\right)$, $0 = \frac{dh}{dx}(0) = c_2\sqrt{\lambda}$, $\lambda > 0$ so $c_2 = 0$

$\Rightarrow$ $\frac{dh}{dx} = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda} x)$, $0 = \frac{dh}{dx}(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda} L)$ want $c_1 \neq 0$ so $\Rightarrow$ $\sin(\sqrt{\lambda} L) = 0$

$\Rightarrow$ $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and $\varphi_n(x)$ (or $h_n(x)$) $= \cos\left(\frac{n\pi x}{L}\right)$

so $u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

and $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$, $A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$

2) $\begin{cases} u_t = \kappa u_{xx} \\ u(-L,t) = u(L,t) \quad , \quad u_x(-L,t) = u_x(L,t) \\ u(x,0) = f(x) \end{cases}$

Let $u(x,t) = h(x)g(t) \Rightarrow h(x)\frac{dg}{dt} = \kappa g(t)\frac{d^2h}{dx^2} \Rightarrow \frac{1}{\kappa}\frac{1}{g}\frac{dg}{dt} = \frac{1}{h(x)}\frac{d^2h}{dx^2} = -\lambda$

$\Rightarrow$ ① $\frac{dg}{dt} = -\kappa\lambda g$ & ② $\begin{cases} \frac{d^2h}{dx^2} + \lambda h = 0 \\ h(-L) = h(L) \\ \frac{dh}{dx}(-L) = \frac{dh}{dx}(L) \end{cases}$ ① $\Rightarrow g(t) = c\, e^{-\kappa\lambda t}$

② $\frac{d^2h}{dx^2} + \lambda h = 0 \Rightarrow r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda}$

Case: $\lambda = 0$ $\Rightarrow r = 0,0$ $\quad h(x) = c_1 + c_2 x \quad h(-L) = h(L) \Rightarrow c_1 + c_2 L = c_1 - c_2 L \Rightarrow 2c_2 L = 0$ so $c_2 = 0$

so $h(x) = c_1 \quad \frac{dh}{dx} = 0 \quad \frac{dh}{dx}(-L) = 0$ & $\frac{dh}{dx}(L) = 0$ so 2nd B.C. satisfied

Case: $\lambda < 0$ let $-\lambda = s^2 > 0$ so $h(x) = c_1 e^{sx} + c_2 e^{-sx}$ & $\frac{dh}{dx} = s(c_1 e^{sx} - c_2 e^{-sx})$

$\begin{cases} h(-L) = c_1 e^{-sL} + c_2 e^{sL} \\ h(L) = c_1 e^{sL} + c_2 e^{-sL} \end{cases}$ $\begin{cases} \frac{dh}{dx}(-L) = s(c_1 e^{-sL} - c_2 e^{sL}) \\ \frac{dh}{dx}(L) = s(c_1 e^{sL} - c_2 e^{-sL}) \end{cases}$

$\begin{cases} (c_1 - c_2)e^{-sL} = (c_1 - c_2)e^{sL} \\ (c_1 - c_2)e^{sL} = (c_1 - c_2)e^{-sL} \end{cases}$ $\Rightarrow$ either $c_1 = c_2$ or $e^{-sL} = e^{sL}$ not possible for $s \neq 0$

so $c_1 = c_2$ then from either B.C. $\Rightarrow e^{-sL} = e^{sL}$ again, not possible again
$\Rightarrow c_1 = c_2 = 0$

Case: $\lambda > 0$: $\Rightarrow r = \pm i\sqrt{\lambda}$ so $h(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

$\frac{dh}{dx} = \sqrt{\lambda}(c_2\cos(\sqrt{\lambda}x) - c_1\sin(\sqrt{\lambda}x))$

$h(-L) = c_1\cos(\sqrt{\lambda}L) - c_2\sin(\sqrt{\lambda}L)$ $\quad h(-L) = h(L) \Rightarrow 2c_2\sin(\sqrt{\lambda}L) = 0$ or $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

$h(L) = c_1\cos(\sqrt{\lambda}L) + c_2\sin(\sqrt{\lambda}L)$

$\frac{dh}{dx}(-L) = \sqrt{\lambda}(c_2\cos(\sqrt{\lambda}L) + c_1\sin(\sqrt{\lambda}L))$ $\quad \frac{dh}{dx}(-L) = \frac{dh}{dx}(L) \Rightarrow 2c_1\sin(\sqrt{\lambda}L) = 0$ or

$\frac{dh}{dx}(L) = \sqrt{\lambda}(c_2\cos(\sqrt{\lambda}L) - c_1\sin(\sqrt{\lambda}L))$

so $\varphi_n(x)$ (or $h_n(x)$) $= \cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi x}{L}\right)$

$$u(x,t) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right)e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t} + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right)e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t}$$

$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

3) $\begin{cases} u_t = k u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$

let $u(x,t) = h(x) g(t) \Rightarrow h(x) \frac{dg}{dt} = k g(t) \frac{d^2h}{dx^2} \Rightarrow \frac{1}{kg}\frac{dg}{dt} = \frac{1}{h}\frac{d^2h}{dx^2} = -\lambda$

$\Rightarrow$ ① $\frac{dg}{dt} = -\lambda k g$ and ② $\begin{cases} \frac{d^2h}{dx^2} + \lambda h = 0 \\ h(0) = h(L) = 0 \end{cases}$ ① $\Rightarrow g(t) = c e^{-k\lambda t}$

② $\frac{d^2h}{dx^2} + \lambda h = 0$ $r = \pm\sqrt{-\lambda}$

Case: $\lambda = 0 \Rightarrow r = 0, 0 \Rightarrow h(x) = c_1 + c_2 x$ $0 = h(0) = c_1 \Rightarrow c_1 = 0$, $0 = h(L) = c_2 L \Rightarrow c_2 = 0$

Case: $\lambda < 0$ set $-\lambda = s^2 > 0$ so ~~$h(x) = c_1 e^{s}$~~ $h(x) = c_1 e^{sx} + c_2 e^{-sx}$

$0 = h(0) = c_1 + c_2 \Rightarrow c_1 = -c_2$ so $h(x) = c_1(e^{sx} - e^{-sx})$

$0 = h(L) = c_1(e^{sL} - e^{-sL})$ want $c_1 \neq 0$ so $\Rightarrow e^{sL} = e^{-sL} \Rightarrow s = 0$ contradiction

$\Rightarrow c_1 = 0$ and so $c_2 = 0$

Case: $\lambda > 0$ so $r = \pm i\sqrt{\lambda}$ so $h(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$

$0 = h(0) = c_1$ so $h(x) = c_2 \sin(\sqrt{\lambda} x)$ and $0 = h(L) = c_2 \sin(\sqrt{\lambda} L) \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$

and $\varphi_n(x)$ (or $h_n(x)$) $= \sin\left(\frac{n\pi x}{L}\right)$

so $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

and $B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

4) $\Delta u = 0$

$$\begin{cases} u(0,y) = u(L,y) = u(x,H) = 0 \\ u(x,0) = f(x) \end{cases}$$

let $u(x,y) = h(x)g(y) \Rightarrow \frac{d^2h}{dx^2}g(y) + h(x)\frac{d^2g}{dy^2} = 0 \Rightarrow -\frac{1}{h(x)}\frac{d^2h}{dx^2} = \frac{1}{g}\frac{d^2g}{dy^2} = \lambda$

$\Rightarrow$ ① $\begin{cases} \frac{d^2h}{dx^2} + \lambda h = 0 \\ h(0) = h(L) = 0 \end{cases}$ and ② $\begin{cases} \frac{d^2g}{dy^2} - \lambda g = 0 \\ g(H) = 0 \end{cases}$

① $\frac{d^2h}{dx^2} + \lambda h = 0$ $\quad r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda}$

<u>case: $\lambda = 0$</u> : $\Rightarrow r = 0, 0 \Rightarrow h(x) = c_1 + c_2 x$, $0 = h(0) = c_1 \Rightarrow c_1 = 0$ and $0 = h(L) = c_2 L \Rightarrow c_2 = 0$

<u>case: $\lambda < 0$</u> : set $-\lambda = s^2 > 0$ so $h(x) = c_1 e^{sx} + c_2 e^{-sx}$

$0 = h(0) = c_1 + c_2 \Rightarrow c_2 = -c_1$ so $h(x) = c_1(e^{sx} - e^{-sx})$

so $0 = h(L) = c_1(e^{sL} - e^{-sL})$ and $c_1 \neq 0 \Rightarrow e^{sL} = e^{-sL} \Rightarrow s = 0$ contradiction

$\Rightarrow c_1 = c_2 = 0$.

<u>case: $\lambda > 0$</u> so $r = \pm i\sqrt{\lambda}$ so $h(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$, $0 = h(0) = c_1$ so $h(x) = c_2\sin(\sqrt{\lambda}x)$

$0 = h(L) = c_2\sin(\sqrt{\lambda}L) \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$ & $\varphi_n(x)$ (or $h_n(x)$) $= \sin\left(\frac{n\pi x}{L}\right)$

② $\frac{d^2g}{dy^2} - \lambda g = 0$ $\quad r^2 = \lambda \Rightarrow r = \pm\sqrt{\lambda}$

<u>case: $\lambda = 0$</u> : $r = 0, 0$ so $g(y) = c_1 + c_2 y$ ~~$g(H)=0$~~ $\Rightarrow 0 = g(H) = c_1 + c_2 H \Rightarrow c_1 = -c_2 H$

so $g(y) = c_2(y - H)$

<u>case: $\lambda > 0$</u> $r = \pm\sqrt{\lambda}$ and $g(y) = c_1 e^{\sqrt{\lambda}y} + c_2 e^{-\sqrt{\lambda}y}$ $\quad 0 = g(H) = c_1 e^{\sqrt{\lambda}H} + c_2 e^{-\sqrt{\lambda}H}$

$\Rightarrow c_1 e^{\sqrt{\lambda}H} = -c_2 e^{-\sqrt{\lambda}H} \Rightarrow c_2 = -c_1 e^{2\sqrt{\lambda}H} \Rightarrow g(y) = c_1(e^{\sqrt{\lambda}y} - e^{2\sqrt{\lambda}H}e^{-\sqrt{\lambda}y})$

$= 2c_1\sinh(\sqrt{\lambda}(y - H))$

so $g_n(y) = \sinh\left(\frac{n\pi}{L}(y - H)\right)$ recall λ_n from ①

<u>case: $\lambda < 0$</u> ~~$r = \pm i\sqrt{|\lambda|} \Rightarrow g(y) = c_1\cos$~~ (not poss as $\lambda_n > 0$ from ①

so $u(x,y) = B_0(y - H) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi}{L}(y - H)\right)$

& $B_0 = \frac{1}{L}\int_0^L f(x)\,dx$, $B_n = \frac{2}{L}\frac{1}{\sinh\left(\frac{-n\pi H}{L}\right)}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

5.) $f(x) = \begin{cases} 0 & x < \frac{L}{2} \\ x & x > \frac{L}{2} \end{cases}$

$$F.S.(f) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

so $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2L}\int_{\frac{L}{2}}^{L} x\,dx = \frac{1}{2L}\frac{x^2}{2}\Big|_{\frac{L}{2}}^{L} = \frac{1}{4L}\left(L^2 - \frac{L^2}{4}\right) = \frac{3L^2}{16L} = \frac{3L}{16}$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{\frac{L}{2}}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left(\frac{L}{n\pi} x \sin\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L} - \frac{L}{n\pi}\int_{\frac{L}{2}}^{L}\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= \frac{1}{L}\left(\frac{L}{n\pi}\left(-\sin\left(\frac{n\pi}{2}\right)\right) + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L}\right) = \frac{1}{L}\left(\frac{-L}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}(-1)^n - \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{L}{n^2\pi^2}(-1)^n - \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{L}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{\frac{L}{2}}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left(-\frac{L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L} + \frac{L}{n\pi}\int_{\frac{L}{2}}^{L}\cos\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= \frac{1}{L}\left(\frac{L^2}{n\pi}(-1)^{n+1} + \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L}\right) = \frac{L}{n\pi}(-1)^{n+1} + \frac{L}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

6.) $f(x) = \begin{cases} -1 & x < \frac{L}{2} \\ 1 & x > \frac{L}{2} \end{cases}$, $F.S.(f) = \sum_{n=0}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2L}\left(\int_{-L}^{\frac{L}{2}} -1\,dx + \int_{\frac{L}{2}}^{L} 1\,dx\right) = \frac{1}{2L}\left(-x\Big|_{-L}^{\frac{L}{2}} + x\Big|_{\frac{L}{2}}^{L}\right) = \frac{1}{2L}\left(-\frac{L}{2} - L + L - \frac{L}{2}\right) = \frac{-L}{2L} = \frac{-1}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left(-\int_{-L}^{\frac{L}{2}}\cos\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^{L}\cos\left(\frac{n\pi x}{L}\right)dx\right) = \frac{1}{L}\left(-\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{\frac{L}{2}} + \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L}\right)$$

$$= \frac{1}{n\pi}\left(-\sin\left(\frac{n\pi}{2}\right) + -\sin\left(\frac{n\pi}{2}\right)\right) = \frac{-2\sin\left(\frac{n\pi}{2}\right)}{n\pi}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left(-\int_{-L}^{\frac{L}{2}}\sin\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^{L}\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= \frac{1}{L}\left(\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{\frac{L}{2}} - \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L}\right) = \frac{1}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) + (-1)^{n+1} + (-1)^{n+1} + \cos\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) + (-1)^{n+1}\right)$$

7.) $u_t = k u_{xx}$

$$\begin{cases} u(-L,t) = T_1(t), \quad u_x(L,t) = T_2(t) \quad (*) \\ u(x,0) = f(x) \end{cases}$$

s'pose u_1 and u_2 both solve $*$ and $u_1 \neq u_2$. then let $w = u_1 - u_2$

and then w solves the following $\begin{cases} w_t = k w_{xx} \\ w(-L,t) = w_x(L,t) = 0 \quad (**) \\ w(x,0) = 0 \end{cases}$

then consider $E(t) = \int_{-L}^{L} w^2(x,t)\,dx$. notice $E(0) = \int_{-L}^{L} w(x,0)^2\,dx = 0$

and since $w^2 \geq 0$ $E(t) \geq 0$. Then $\frac{dE}{dt} = 2\int_{-L}^{L} w w_t\,dx \overset{\text{by using } **}{=} 2\int_{-L}^{L} k w w_{xx}\,dx$

then using by parts $u = w$ $dv = w_{xx}dx$ we get
$du = w_x dx$ $v = w_x$

$\frac{dE}{dt} = 2k\left(w w_x \Big|_{-L}^{L} - \int_{-L}^{L} w_x^2\,dx \right) = -2k\int_{-L}^{L} w_x^2\,dx$ by $**$.

since $k > 0$, $w_x^2 \geq 0 \Rightarrow \frac{dE}{dt} \leq 0$ so $E \searrow$ and $E(t) \geq 0$ and $E(0) = 0 \Rightarrow E = 0$ all t

$\Rightarrow w(x,t) = 0$ for all t, x so $u_1 = u_2$ thus $*$ has unique soln.

8.) $u_t = k u_{xx}$

$$\begin{cases} u(-L,t) = T_1(t), \quad u(L,t) = T_2(t) \quad (*) \\ u(x,0) = f(x) \end{cases}$$

s'pose u_1 and u_2 both solve $*$ and $u_1 \neq u_2$, then let $w = u_1 - u_2$ and then w solves

the following $\begin{cases} w_t = k w_{xx} \\ w(-L,t) = w(L,t) = 0 \quad (**) \\ w(x,0) = 0 \end{cases}$, then consider $E(t) = \int_{-L}^{L} w(x,t)^2\,dx$

notice $E(0) = \int_{-L}^{L} w(x,0)^2\,dx = 0$

and since $w^2 \geq 0 \Rightarrow E(t) \geq 0$. Then $\frac{dE}{dt} = 2\int_{-L}^{L} w w_t\,dx \overset{\text{by using } **}{=} 2k\int_{-L}^{L} w w_{xx}\,dx$ by using by parts $u = w$ $du = dx$ $dv = w_{xx}dx$ $v = w_x$

we get $\frac{dE}{dt} = 2k\left(w w_x \Big|_{-L}^{L} - \int_{-L}^{L} w_x^2\,dx \right) = -2k\int_{-L}^{L} w_x^2\,dx$

since $k > 0$, $w_x^2 \geq 0 \Rightarrow \frac{dE}{dt} \leq 0$ so $E \searrow$ and $E \geq 0$ and $E(0) = 0 \Rightarrow E = 0$ for all t

$\Rightarrow w(x,t) = 0$ for all x, t so $u_1 = u_2$ thus $*$ has unique soln.

4.) $u(x,t) = \frac{1}{\sqrt{4\kappa\pi t}} \int_{\mathbb{R}} f(y) e^{\frac{-(x-y)^2}{4\kappa t}} \, dy = (4\kappa\pi t)^{-\frac{1}{2}} \int_{\mathbb{R}} f(y)\, e^{\frac{-(x-y)^2}{4\kappa t}} \, dy$

$$u_t = -\frac{1}{2}(4\kappa\pi t)^{-\frac{3}{2}} 4\kappa\pi \int_{\mathbb{R}} f(y) e^{\frac{-(x-y)^2}{4\kappa t}} dy + (4\kappa\pi t)^{-\frac{1}{2}} \int_{\mathbb{R}} f(y)\, e^{\frac{-(x-y)^2}{4\kappa t}} \frac{(x-y)^2}{4\kappa t^2} \, dy$$

$$= \frac{1}{\sqrt{4\kappa\pi t}} \left(-\frac{2\kappa\pi}{4\kappa\pi t} \int_{\mathbb{R}} f(y) e^{\frac{-(x-y)^2}{4\kappa t}} dy + \frac{1}{4\kappa t^2} \int_{\mathbb{R}} f(y)\, e^{\frac{-(x-y)^2}{4\kappa t}} (x-y)^2 dy \right)$$

$$= \frac{1}{\sqrt{4\kappa\pi t}} \left(\frac{-1}{2t} \int_{\mathbb{R}} f(y)\, e^{\frac{-(x-y)^2}{4\kappa t}} dy + \frac{1}{4\kappa t^2} \int_{\mathbb{R}} f(y)\, e^{\frac{-(x-y)^2}{4\kappa t}} (x-y)^2 dy \right)$$

$$u_x = \frac{1}{\sqrt{4\kappa\pi t}} \int_{\mathbb{R}} f(y)\, e^{\frac{-(x-y)^2}{4\kappa t}} \left(\frac{-2(x-y)}{(4\kappa t)} \right) dy = \frac{-2}{4\kappa t \sqrt{4\kappa\pi t}} \int_{\mathbb{R}} f(y)\, e^{\frac{-(x-y)^2}{4\kappa t}} (x-y) \, dy$$

$$u_{xx} = \frac{+4}{(4\kappa t)^2 \sqrt{4\kappa\pi t}} \int_{\mathbb{R}} f(y)\, e^{\frac{-(x-y)^2}{4\kappa t}} (x-y)^2 dy - \frac{2}{4\kappa t\sqrt{4\kappa\pi t}} \int_{\mathbb{R}} f(y)\, e^{\frac{-(x-y)^2}{4\kappa t}} dy$$

$$= \frac{1}{\sqrt{4\kappa\pi t}} \left(\frac{4}{(4\kappa t)^2} \int_{\mathbb{R}} f(y)\, e^{\frac{-(x-y)^2}{4\kappa t}} (x-y)^2 dy - \frac{2}{4\kappa t} \int_{\mathbb{R}} f(y)\, e^{\frac{-(x-y)^2}{4\kappa t}} dy \right)$$

$$\Rightarrow \kappa u_{xx} = \frac{1}{\sqrt{4\kappa\pi t}} \left(\frac{1}{4\kappa t^2} \int_{\mathbb{R}} f(y)\, e^{\frac{-(x-y)^2}{4\kappa t}} (x-y)^2 dy - \frac{1}{2t} \int_{\mathbb{R}} f(y)\, e^{\frac{-(x-y)^2}{4\kappa t}} dy \right) = u_t$$