1.) $\begin{cases} \Delta v + \lambda v = 0 \\ v|_{\partial\Omega} = 0 \end{cases}$

let $\lambda_1 \xrightarrow{\text{associated}}$ to φ_1 and $\lambda_2 \xrightarrow{\text{associated}}$ to φ_2 for $\lambda_1 \neq \lambda_2$ then they both solve

$\begin{cases} \Delta\varphi_1 = -\lambda_1\varphi_1 \\ \varphi_1|_{\partial\Omega} = 0 \end{cases}$ and $\begin{cases} \Delta\varphi_2 = -\lambda_2\varphi_2 \\ \varphi_2|_{\partial\Omega} = 0 \end{cases}$ $\Rightarrow \begin{cases} \varphi_2\Delta\varphi_1 = -\lambda_1\varphi_1\varphi_2 \\ \varphi_1\Delta\varphi_2 = -\lambda_2\varphi_1\varphi_2 \end{cases}$

$\Rightarrow (\lambda_1-\lambda_2)\varphi_1\varphi_2 = \varphi_1\Delta\varphi_2 - \varphi_2\Delta\varphi_1 \Rightarrow (\lambda_1-\lambda_2)\iint_\Omega \varphi_1\varphi_2\, dA = \iint_\Omega(\varphi_1\Delta\varphi_2 - \varphi_2\Delta\varphi_1)\, dA$

$= \int_{\partial\Omega}(\varphi_1\nabla\varphi_2 - \varphi_2\nabla\varphi_1)\cdot\vec{n}\, ds$ by Green's Formula, but $\varphi_1|_{\partial\Omega} = 0$

$\Rightarrow \int_{\partial\Omega}\varphi_1\nabla\varphi_2\cdot\vec{n}\, ds = 0$ and similarly $\varphi_2|_{\partial\Omega} = 0 \Rightarrow \int_{\partial\Omega}\varphi_2\nabla\varphi_1\cdot\vec{n}\, ds = 0$

$\Rightarrow (\lambda_1-\lambda_2)\iint_\Omega \varphi_1\varphi_2\, dA = 0$ since $\lambda_1 \neq \lambda_2 \Rightarrow \iint_\Omega \varphi_1\varphi_2\, dA = 0$ so φ_1, φ_2 are orthogonal

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2.) $\begin{cases} \Delta v + \lambda v = 0 \\ v|_{\partial\Omega} = 0 \end{cases}$

let $\lambda \xrightarrow{\text{assoc.}}$ to φ. Then by conjugation know $\bar{\lambda} \to \bar{\varphi}$ and they both solve

$\begin{cases} \Delta\varphi = -\lambda\varphi \\ \varphi|_{\partial\Omega} = 0 \end{cases}$ and $\begin{cases} \Delta\bar{\varphi} = -\bar{\lambda}\bar{\varphi} \\ \bar{\varphi}|_{\partial\Omega} = 0 \end{cases}$ $\Rightarrow \begin{cases} \bar{\varphi}\Delta\varphi = -\lambda\varphi\bar{\varphi} \\ \varphi\Delta\bar{\varphi} = -\bar{\lambda}\bar{\varphi}\varphi \end{cases}$

$\Rightarrow (\lambda-\bar{\lambda})\varphi\bar{\varphi} = \varphi\Delta\bar{\varphi} - \bar{\varphi}\Delta\varphi \Rightarrow (\lambda-\bar{\lambda})\iint_\Omega |\varphi|^2\, dA = \iint_\Omega(\varphi\Delta\bar{\varphi} - \bar{\varphi}\Delta\varphi)\, dA$

$= \int_{\partial\Omega}(\varphi\nabla\bar{\varphi} - \bar{\varphi}\nabla\varphi)\cdot\vec{n}\, ds$ by Green's Formula. but $\varphi|_{\partial\Omega} = 0$

$\Rightarrow \int_{\partial\Omega}\varphi\nabla\bar{\varphi}\cdot\vec{n}\, ds = 0$ and $\bar{\varphi}|_{\partial\Omega} = 0 \Rightarrow \int_{\partial\Omega}\bar{\varphi}\nabla\varphi\cdot\vec{n}\, ds = 0$

so $(\lambda-\bar{\lambda})\iint_\Omega|\varphi|^2\, dA = 0$ but the integral is pos. $\Rightarrow \lambda-\bar{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda}$ so λ real.

3.) $$\begin{cases} u_{tt} = c^2 \Delta u \\ u(a,\theta,t) = 0 \\ u(r,\theta,0) = \alpha(r,\theta) \ , \ u_t(r,\theta,0) = \beta(r,\theta) \end{cases}$$

let $u(r,\theta,t) = h(t)\,v(r,\theta) \Rightarrow \frac{d^2h}{dt^2} v = c^2 h \Delta v \Rightarrow \frac{1}{c^2}\frac{1}{h}\frac{d^2h}{dt^2} = \frac{\Delta v}{v} = -\lambda$

so set $\frac{d^2h}{dt^2} + c^2\lambda h = 0$ & $\begin{cases} \Delta v + \lambda v = 0 \\ v(a,\theta) = 0 \end{cases}$

now let $v(r,\theta) = f(r)\,g(\theta) \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) g(\theta) + \frac{1}{r^2} f \frac{d^2g}{d\theta^2} = -\lambda f g$

$\Rightarrow r\frac{d}{dr}\left(r\frac{df}{dr}\right)\frac{1}{f} + \frac{1}{g}\frac{d^2g}{d\theta^2} = -r^2\lambda \Rightarrow \frac{1}{f} r \frac{d}{dr}\left(r\frac{df}{dr}\right) + r^2\lambda = -\frac{1}{g}\frac{d^2g}{d\theta^2} = \mu$

$\Rightarrow \begin{cases} r^2\frac{d^2f}{dr^2} + r\frac{df}{dr} + (\lambda r^2 - \mu) f = 0 \\ f(a) = 0 \ \& \ |f(0)| < \infty \end{cases}$ & $\begin{cases} \frac{d^2g}{d\theta^2} + \mu g = 0 \\ g(-\pi) = g(\pi) \\ \frac{dg}{d\theta}(-\pi) = \frac{dg}{d\theta}(\pi) \end{cases}$

comes for free

so in eq w/ g's $\mu_m = m^2$ & $g_m(\theta) = \sin(m\theta)\ ,\ \cos(m\theta)$

then for eq w/ f's set $z = \sqrt{\lambda}\, r \Rightarrow z^2\frac{d^2f}{dz^2} + z\frac{df}{dz} + (z^2 - m^2) f = 0$

is Bessel's diff. eq. & sol'n's are $f(z) = c_1 J_m(z) + c_2 Y_m(z)$ w/ $J_m(z) \sim \begin{cases} z^m & m>0 \\ 1 & m=0 \end{cases}$ $\quad Y_m(z) \sim \begin{cases} z^{-m} & m>0 \\ -\log z & m=0 \end{cases}$

So need $|f(0)| < \infty$ & $Y_m(z)$ bad sol'n at $z=0$ so $f(z) = J_m(z)$

need $f(a) = 0 \Rightarrow$ let $J_m(z) = J_m(\sqrt{\lambda_{mn}}\, r)$

$\Rightarrow \sqrt{\lambda_{mn}}\, a$ to be a zero to $J_m(z)$ and all positive so $\lambda_{mn} > 0$

$\Rightarrow$ in eq for h's get $h_{mn}(t) = \cos(c\sqrt{\lambda_{mn}}\, t)\ ,\ \sin(c\sqrt{\lambda_{mn}}\, t)$

so $$u(r,\theta,t) = \sum_{\substack{m=1\\n=0}}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\sin(m\theta)\cos(c\sqrt{\lambda_{mn}}\, t) + \sum_{\substack{m=1\\n=0}}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\sin(m\theta)\sin(c\sqrt{\lambda_{mn}}\, t)$$
$$+ \sum_{\substack{m=0\\n=0}}^{\infty} C_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\cos(m\theta)\cos(c\sqrt{\lambda_{mn}}\, t) + \sum_{\substack{m=0\\n=0}}^{\infty} D_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\cos(m\theta)\sin(c\sqrt{\lambda_{mn}}\, t)$$

4.) $\begin{cases} u_{tt} = c^2 \Delta u \\ u_r(a,\theta,t) = 0 \\ u(r,\theta,0) = \alpha(r,\theta) \quad , \quad u_t(r,\theta,0) = \beta(r,\theta) \end{cases}$

let $u(r,\theta,t) = h(t)\,v(r,\theta) \Rightarrow \dfrac{d^2h}{dt^2} v = c^2 h \Delta v \Rightarrow \dfrac{1}{c^2}\dfrac{1}{h}\dfrac{d^2h}{dt^2} = \dfrac{\Delta v}{v} = -\lambda$

so get $\dfrac{d^2h}{dt^2} + c^2\lambda h = 0$ and $\begin{cases} \Delta v + \lambda v \\ v_r(a,\theta) = 0 \end{cases}$ let $v(r,\theta) = f(r)\,g(\theta)$

$\Rightarrow \dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{df}{dr}\right)g(\theta) + \dfrac{1}{r^2}f(r)\dfrac{d^2g}{d\theta^2} = -\lambda fg \Rightarrow r\dfrac{1}{f}\dfrac{d}{dr}\left(r\dfrac{df}{dr}\right) + \dfrac{1}{g}\dfrac{d^2g}{d\theta^2} = -\lambda r^2$

$\Rightarrow r\dfrac{1}{f}\dfrac{d}{dr}\left(r\dfrac{df}{dr}\right) + \lambda r^2 = -\dfrac{1}{g}\dfrac{d^2g}{d\theta^2} = \mu$

get $\begin{cases} r^2\dfrac{d^2f}{dr^2} + r\dfrac{df}{dr} + (\lambda r^2 - \mu)f = 0 \\ \dfrac{df}{dr}(a) = 0 \quad , \quad |f(0)| < \infty \end{cases}$ and $\begin{cases} \dfrac{d^2g}{d\theta^2} + \mu g = 0 \\ g(-\pi) = g(\pi) \\ \dfrac{dg}{d\theta}(-\pi) = \dfrac{dg}{d\theta}(\pi) \end{cases}$

in eq's w/ g's have $\mu_m = m^2$, and $g_m(\theta) = \cos(m\theta)\ ,\ \sin(m\theta)$

in eq's w/ f's set $z = \sqrt{\lambda}\,r \Rightarrow z^2\dfrac{d^2f}{dz^2} + z\dfrac{df}{dz} + (z^2 - m^2)f = 0$

which is Bessel's Diff. Eq. and sol'n's are $J_m(z)$, $Y_m(z)$ w/

$J_m(z) \sim \begin{cases} z^m & m>0 \\ 1 & m=0 \end{cases}$

$Y_m(z) \sim \begin{cases} z^{-m} & m>0 \\ -\log z & m=0 \end{cases}$

need $|f(0)| < \infty$ and $Y_m(z)$ is bad near zero so $f(z) = J_m(z)$

and so $f(r) = J_m(\sqrt{\lambda_{mn}}\,r)$ ~~and~~ want $\dfrac{df}{dr}(a) = 0$ so $\dfrac{dJ_m}{dr}(\sqrt{\lambda_{mn}}\,a) = 0$ need $\sqrt{\lambda_{mn}}\,a$ to be a zero of the derivative of $J_m(z)$. and $\lambda_{mn} > 0$

so in eq's w/ h get $h_{mn}(t) = \cos(c\sqrt{\lambda_{mn}}\,t)\ ,\ \sin(c\sqrt{\lambda_{mn}}\,t)$

so $u(r,\theta,t) = \displaystyle\sum_{\substack{m=1\\n=0}}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)\cos(c\sqrt{\lambda_{mn}}\,t) + \sum_{\substack{m=1\\n=0}}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)\sin(c\sqrt{\lambda_{mn}}\,t)$

$+ \displaystyle\sum_{\substack{m=0\\n=0}}^{\infty} C_{mn} J_m(\sqrt{\lambda_{mn}}\,r)\cos(m\theta)\cos(c\sqrt{\lambda_{mn}}\,t) + \sum_{\substack{m=0\\n=0}}^{\infty} D_{mn} J_m(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)\cos(c\sqrt{\lambda_{mn}}\,t)$

5.) $(x^6+x^2)\frac{d^2y}{dx^2} + (x^7+x)\frac{dy}{dx} + (6x^5-4)y = 0$

want to guess soln's of form $y = x^p$ near zero so can throw away x^6 in $\frac{d^2y}{dx^2}$, x^7 in $\frac{dy}{dx}$ and $6x^5$ in y terms otherwise powers won't match.

$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y \approx 0$ so $y = x^p$ and $y' = px^{p-1}$ and $y'' = p(p-1)x^{p-2}$

plugging in gives $p(p-1)x^p + px^p - 4x^p \approx 0 \Rightarrow p(p-1) + p - 4 = 0$

$p^2 - p + p - 4 = 0$ so $p^2 - 4 = 0$ so $p = \pm 2$

so $y \approx c_1 x^2 + c_2 x^{-2} + \ldots \leftarrow$ lower terms we don't know.

6.) $(x^6+x^2)\frac{d^2y}{dx^2} + \left(x^5 + \frac{3}{16}\right)y = 0$

want to guess soln's of form $y = x^p$ near zero so can throw away x^6 in y'' and x^5 in y terms otherwise powers won't match

$\Rightarrow x^2\frac{d^2y}{dx^2} + \frac{3}{16}y \approx 0$ so $y = x^p$ and $y' = px^{p-1}$ and $y'' = p(p-1)x^{p-2}$

plugging in gives $p(p-1)x^p + \frac{3}{16}x^p \approx 0 \Rightarrow p(p-1) + \frac{3}{16} = 0$

$\Rightarrow p^2 - p + \frac{3}{16} = 0$ or $16p^2 - 16p + 3 = 0$ so $p = \frac{16 \pm \sqrt{256 - 192}}{32} = \frac{1}{2} \pm \frac{8}{32} = \frac{1}{2} \pm \frac{1}{4}$

so $p = \frac{3}{4}, \frac{1}{4}$ so $y \approx c_1 x^{\frac{3}{4}} + c_2 x^{\frac{1}{4}} + \ldots \leftarrow$ lower terms we don't know

7.) $\begin{cases} u_t = \kappa u_{xx} + Q(x,t) \\ u_x(0,t) = A(t), \quad u(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$

The associated homogeneous problem is $\begin{cases} \frac{d^2h}{dx^2} + \lambda h = 0 \\ \frac{dh}{dx}(0) = h(L) = 0 \end{cases}$ know $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$ $n = 1, 2, \ldots$ and sol'ns $\varphi_n(x) = \cos\left(\frac{(2n-1)\pi x}{2L}\right)$

$\frac{d\varphi_n}{dx} = -\frac{(2n-1)\pi}{2L} \sin\left(\frac{(2n-1)\pi x}{2L}\right)$

s'pose $u(x,t) = \sum_{n=0}^{\infty} b_n(t)\varphi_n(x)$ then $b_n(t) = \dfrac{\int_0^L u\,\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx}$

and s'pose $Q(x,t) = \sum_{n=0}^{\infty} q_n(t)\varphi_n(x)$ then $q_n(t) = \dfrac{\int_0^L Q\,\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx}$

then $u_t = \sum_{n=0}^{\infty} \frac{db_n}{dt}\varphi_n(x) = \kappa u_{xx} + Q$

so $\frac{db_n}{dt} = \dfrac{\kappa\int_0^L u_{xx}\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx} + \dfrac{\int_0^L Q\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx} = \kappa\dfrac{\int_0^L u_{xx}\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx} + q_n(t)$

Using Greens formula $\int_0^L u_{xx}\varphi_n(x)\,dx = \int_0^L u\frac{d^2\varphi_n}{dx^2}\,dx + \left(\varphi_n u_x - u\frac{d\varphi_n}{dx}\right)\Big|_0^L$

$= -\lambda_n\int_0^L u\,\varphi_n\,dx + \left(\varphi_n(L)u_x(L) - u(L)\frac{d\varphi_n}{dx}(L)\right) - \left(\varphi_n(0)u_x(0) - u(0)\frac{d\varphi_n}{dx}(0)\right)$

$= -\lambda_n\int_0^L u\,\varphi_n\,dx + 0 - B(t)(-1)^n - (A(t) - 0) = -\lambda_n\int_0^L u\,\varphi_n\,dx - (A(t) + (-1)^n B(t))$

so $\frac{db_n}{dt} = -\lambda\kappa b_n + q_n(t) - (A(t) + (-1)^n B(t)) \Rightarrow \frac{db_n}{dt} + \lambda_n\kappa b_n = q_n(t) - (A(t) + (-1)^n B(t))$

is a 1st order eq. and integrating fact is $\mu = \exp\left(\int\lambda_n\kappa\,dt\right) = e^{\lambda_n\kappa t}$

$\Rightarrow (b_n e^{\lambda_n\kappa t})' = \{q_n(t) - (A(t) + (-1)^n B(t))\}e^{\lambda_n\kappa t}$

$\Rightarrow b_n e^{\lambda_n\kappa t} = b_n(0) + \int_0^t [q_n(\tau) - (A(\tau) + (-1)^n B(\tau))]e^{\lambda_n\kappa\tau}\,d\tau$

So $b_n(t) = b_n(0)e^{-\lambda_n\kappa t} + e^{-\lambda_n\kappa t}\int_0^t [q_n(\tau) - (A(\tau) + (-1)^n B(\tau))]e^{\lambda_n\kappa\tau}\,d\tau$

so $u(x,t) = \sum_{n=0}^{\infty} b_n(t)\cos\left(\frac{(2n-1)\pi x}{2L}\right)$ w/ $b_n(t)$ and $b_n(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{(2n-1)\pi x}{2L}\right)dx$

8.) $\begin{cases} u_t = k u_{xx} + Q(x,t) \\ u_x(0,t) = A(t), \quad u_x(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$

The associated homogeneous prob is $\begin{cases} \frac{d^2h}{dx^2} + \lambda h = 0 \\ \frac{dh}{dx}(0) = \frac{dh}{dx}(L) = 0 \end{cases}$ know $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, \dots$ and soln's $\varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$

S'pose $u(x,t) = \sum_{n=0}^{\infty} b_n(t)\varphi_n(x)$ then $b_n(t) = \dfrac{\int_0^L u\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx}$

and s'pose $Q(x,t) = \sum_{n=0}^{\infty} q_n(t)\varphi_n(x)$ then $q_n(t) = \dfrac{\int_0^L Q\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx}$

Then $u_t = \sum_{n=0}^{\infty} \frac{db_n}{dt}\varphi_n(x) = k u_{xx} + Q$

so $\dfrac{db_n}{dt} = k\dfrac{\int_0^L u_{xx}\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} + \dfrac{\int_0^L Q\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} = k\dfrac{\int_0^L u_{xx}\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} + q_n(t)$

using Green's formula $\int_0^L u_{xx}\varphi_n\,dx = \int_0^L u\frac{d^2\varphi_n}{dx^2}\,dx + \left(\varphi_n u_x - u\frac{d\varphi_n}{dx}\right)\Big|_0^L$

$= -\lambda_n\int_0^L u\varphi_n\,dx + \left(\varphi_n(L)u_x(L) - u(L)\frac{d\varphi_n}{dx}(L)\right) - \left(\varphi_n(0)u_x(0) - u(0)\frac{d\varphi_n}{dx}(0)\right)$

$= -\lambda_n\int_0^L u\varphi_n\,dx + (-1)^n B(t) - A(t)$ $\Rightarrow$ $\frac{db_n}{dt} = -\lambda_n k b_n + (-1)^n B(t) - A(t) + q_n(t)$

so $\frac{db_n}{dt} + \lambda_n k b_n = q_n(t) + (-1)^n B(t) - A(t)$ is a 1st order eq w/ integrating factor

$\mu = \exp\left(\int \lambda_n k\,dt\right) = e^{\lambda_n k t}$ $\Rightarrow$ $\left(b_n e^{\lambda_n k t}\right)' = \left[q_n(t) + (-1)^n B(t) - A(t)\right]e^{\lambda_n k t}$

$\Rightarrow$ $b_n e^{\lambda_n k t} = b_n(0) + \int_0^t \left[q_n(\tau) + (-1)^n A(\tau) - B(\tau)\right]e^{\lambda_n k \tau}\,d\tau$

so $b_n(t) = b_n(0)e^{-\lambda_n k t} + e^{-\lambda_n k t}\int_0^t \left[q_n(\tau) + (-1)^n A(\tau) - B(\tau)\right]e^{\lambda_n k \tau}\,d\tau$

so $u(x,t) = \sum_{n=0}^{\infty} b_n(t)\cos\left(\frac{n\pi x}{L}\right)$ w/ $b_n(t)$ and $b_n(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

4.) $u(x,y,t) = g(x-b_1 t,\ y-b_2 t) + \int_0^t f(x+(s-t)b_1,\ y+(s-t)b_2,\ s)\,ds$

set $\alpha = x - b_1 t \qquad \gamma = x + (s-t) b_1$

$\beta = y - b_2 t \qquad \delta = y + (s-t) b_2$

then $u(x,y,t) = g(\alpha, \beta) + \int_0^t f(\gamma, \delta, s)\,ds$

so $u_t = \frac{\partial g}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial g}{\partial \beta}\frac{\partial \beta}{dt} + f(x,y,t) + \int_0^t \left(\frac{\partial f}{\partial \gamma}\frac{\partial \gamma}{\partial t} + \frac{\partial f}{\partial \delta}\frac{\partial \delta}{\partial t} \right) ds$

$= -\frac{\partial g}{\partial \alpha} b_1 - \frac{\partial g}{\partial \beta} b_2 + f(x,y,t) - \int_0^t \left(\frac{\partial f}{\partial \gamma} b_1 + \frac{\partial f}{\partial \delta} b_2 \right) ds$

then $\nabla u = \left(\frac{\partial g}{\partial \alpha}\frac{\partial \alpha}{\partial x},\ \frac{\partial g}{\partial \beta}\frac{\partial \beta}{\partial x} \right) + \int_0^t \left(\frac{\partial f}{\partial \gamma}\frac{\partial \gamma}{\partial x},\ \frac{\partial f}{\partial \delta}\frac{\partial \delta}{\partial y} \right) ds$

$= \left(\frac{\partial g}{\partial \alpha},\ \frac{\partial g}{\partial \beta} \right) + \int_0^t \left(\frac{\partial f}{\partial \gamma},\ \frac{\partial f}{\partial \delta} \right) ds$

so $\vec{b} \cdot \nabla u = b_1 \frac{\partial g}{\partial \alpha} + b_2 \frac{\partial g}{\partial \beta} + \int_0^t \left(b_1 \frac{\partial f}{\partial \gamma} + b_2 \frac{\partial f}{\partial \delta} \right) ds$

so $u_t + \vec{b} \cdot \nabla u = -\frac{\partial g}{\partial \alpha} b_1 - \frac{\partial g}{\partial \beta} b_2 - \int_0^t \left(\frac{\partial f}{\partial \gamma} b_1 + \frac{\partial f}{\partial \delta} b_2 \right) ds + f(x,y,t)$

$+ \frac{\partial g}{\partial \alpha} b_1 + \frac{\partial g}{\partial \beta} b_2 + \int_0^t \left(\frac{\partial f}{\partial \gamma} b_1 + \frac{\partial f}{\partial \delta} b_2 \right)$

$= f(x,y,t)$