1.) $\begin{cases} u_t = \kappa u_{xx} \\ u(-L,t) = u(L,t), \quad u_x(-L,t) = u_x(L,t) \\ u(x,0) = f(x) \end{cases}$

set $u(x,t) = h(x)g(t) \Rightarrow h\frac{dg}{dt} = \kappa g \frac{d^2h}{dx^2} \Rightarrow \frac{1}{\kappa}\frac{1}{g}\frac{dg}{dt} = \frac{1}{h}\frac{d^2h}{dx^2} = -\lambda$

$\Rightarrow$ ① $\frac{dg}{dt} = -\kappa\lambda g$ & ② $\frac{d^2h}{dx^2} + \lambda h = 0$
$h(-L) = h(L)$
$\frac{dh}{dx}(-L) = \frac{dh}{dx}(L)$

① gives $g(t) = e^{-\kappa\lambda t}$ & for ② have char. eq. $r^2 + \lambda = 0$

$\Rightarrow r^2 = \pm\sqrt{-\lambda}$ if $\lambda = 0 \Rightarrow r = 0$ & $h(x) = c_1 x + c_2$ $\Rightarrow h(-L) = -c_1 L + c_2 = h(L) = c_1 L + c_2$

$\Rightarrow c_1 L = -c_1 L \Rightarrow c_1 = 0$ & $h'(x) = 0$ so $h'(-L) = h'(L)$ always satisfied.

if $\lambda < 0 \Rightarrow -\lambda = s^2 > 0$ so $r = \pm s$ so $h(x) = c_1 e^{sx} + c_2 e^{-sx}$

& $h'(x) = s(c_1 e^{sx} - c_2 e^{-sx})$ & $h(L) = c_1 e^{sL} + c_2 e^{-sL} = h(-L) = c_1 e^{-sL} + c_2 e^{sL}$

$h'(L) = s(c_1 e^{sL} - c_2 e^{-sL}) = h'(-L) = s(c_1 e^{-sL} - c_2 e^{sL})$

$\Rightarrow (c_1 - c_2)e^{sL} = (c_1 - c_2)e^{-sL} \Rightarrow (c_1 - c_2)(e^{sL} - e^{-sL}) = 0 \Rightarrow c_1 = c_2$ or $e^{sL} = e^{-sL}$ can't happen

so set $e^{sL} - e^{-sL} = e^{-sL} - e^{sL} \Rightarrow e^{sL} = e^{-sL}$ can't happen $\Rightarrow c_1 = c_2 = 0$

if $\lambda > 0 \Rightarrow r = \pm i\sqrt{\lambda}$ so $h(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$ & $h'(x) = \sqrt{\lambda}(c_2\cos(\sqrt{\lambda}x) - c_1\sin(\sqrt{\lambda}x))$

$h(L) = c_1\cos(\sqrt{\lambda}L) + c_2\sin(\sqrt{\lambda}L) = h(-L) = c_1\cos(\sqrt{\lambda}L) - c_2\sin(\sqrt{\lambda}L)$ (cos is even, sin is odd)

$\Rightarrow \sin(\sqrt{\lambda}L) = 0 \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$ ~~that for~~ and for $h'(-L) = h'(L)$

get $h'(L) = \sqrt{\lambda}(c_2\cos(\sqrt{\lambda}L) - c_1\sin(\sqrt{\lambda}L)) = h'(-L) = \sqrt{\lambda}(c_2\cos(\sqrt{\lambda}L) + c_1\sin(\sqrt{\lambda}L))$

eigen set $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ & eigen fncs $\varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)$

so $u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t}$

& $A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$, $\underset{n\geq 1}{A_n} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$, $B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

2.) $u_t = \kappa u_{xx}$

$$\begin{cases} u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

set $u(x,t) = h(x) g(t)$ $\Rightarrow$ $h \frac{dg}{dt} = \kappa g \frac{d^2h}{dx^2}$ $\Rightarrow$ $\frac{1}{\kappa} \frac{1}{g} \frac{dg}{dt} = \frac{1}{h} \frac{d^2h}{dx^2} = -\lambda$

$\Rightarrow$ ① $\frac{dg}{dt} = -\kappa \lambda g$ ② $\begin{cases} \frac{d^2h}{dx^2} + \lambda h = 0 \\ h(0) = h(L) = 0 \end{cases}$ ① gives $g(t) = e^{-\kappa \lambda t}$

for ② have char. eq. $r^2 + \lambda = 0$ so $r = \pm\sqrt{-\lambda}$: if $\lambda = 0 \Rightarrow r = 0, 0$

& $h(x) = c_1 x + c_2$ $0 = h(0) = c_2 \Rightarrow c_2 = 0$ & $0 = h(L) = c_1 L$ $\Rightarrow$ $c_1 L = 0 \Rightarrow c_1 = 0$

if $\lambda < 0 \Rightarrow -\lambda = -s^2 > 0$ so $r = \pm s$ $\Rightarrow$ $h(x) = c_1 e^{sx} + c_2 e^{-sx}$

$0 = h(0) = c_1 + c_2 \Rightarrow c_2 = -c_1$ $\Rightarrow$ $h(x) = c_1 (e^{sx} - e^{-sx})$ then $0 = h(L) = c_1 (e^{sL} - e^{-sL})$

but $e^{sL} \neq e^{-sL}$ unless $s = 0$ but $\lambda \neq 0$ so can't happen $\Rightarrow c_1 = 0$ so $c_2 = 0$

if $\lambda > 0$ then $r = \pm i\sqrt{\lambda}$ & $h(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$ then

$0 = h(0) = c_1 \Rightarrow c_1 = 0$ then $0 = h(L) = c_2 \sin(\sqrt{\lambda} L)$ $\Rightarrow$ $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ & $h(x) = \sin\left(\frac{n\pi x}{L}\right)$

so $u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\kappa \left(\frac{n\pi}{L}\right)^2 t}$ & $A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

3.) $\begin{cases} L(h) + \lambda \sigma h = 0 \\ h(a) = h(b) = 0 \end{cases}$ and $L(h) = \frac{d}{dx}\left(p \frac{dh}{dx}\right) + qh$

let $\lambda_n \leftrightarrow \varphi_n(x)$ and $\lambda_m \leftrightarrow \varphi_m(x)$ be different eigenvalues associated to the eigenfunctions

then $\begin{cases} L(\varphi_n) + \lambda_n \sigma \varphi_n = 0 \\ L(\varphi_m) + \lambda_m \sigma \varphi_m = 0 \end{cases} \Rightarrow \begin{cases} \varphi_m L(\varphi_n) + \lambda_n \sigma \varphi_n \varphi_m = 0 \\ \varphi_n L(\varphi_m) + \lambda_m \sigma \varphi_n \varphi_m = 0 \end{cases}$

$\Rightarrow \int_a^b \left(\varphi_m L(\varphi_n) - \varphi_n L(\varphi_m)\right) dx = (\lambda_m - \lambda_n) \int_a^b \varphi_n \varphi_m \sigma \, dx$

but by Lagrange's identity $\int_a^b \left(\varphi_m L(\varphi_n) - \varphi_n L(\varphi_m)\right) dx = p\left(\varphi_m \frac{d\varphi_n}{dx} - \varphi_n \frac{d\varphi_m}{dx}\right)\Big|_a^b$

$= p\left(\varphi_m(b) \frac{d\varphi_n}{dx}(b) - \varphi_n(b) \frac{d\varphi_m}{dx}(b)\right) - p\left(\varphi_m(a) \frac{d\varphi_n}{dx}(a) - \varphi_n(a) \frac{d\varphi_m}{dx}(a)\right) = 0$

since $\varphi_m(b) = \varphi_m(a) = \varphi_n(b) = \varphi_n(a) = 0$ since φ_n, φ_m both solve the S-L prob.

so $(\lambda_m - \lambda_n) \int_a^b \varphi_n \varphi_m \sigma \, dx = 0 \Rightarrow \int_a^b \varphi_n \varphi_m \sigma \, dx = 0$ since $\lambda_n \neq \lambda_m$.

4.) $\begin{cases} L(h) + \lambda \sigma h = 0 \\ h(a) = h(b) = 0 \end{cases}$ and $L(h) = \frac{d}{dx}\left(p \frac{dh}{dx}\right) + qh$

let $\lambda \leftrightarrow \varphi(x)$ and $\bar{\lambda} \leftrightarrow \overline{\varphi(x)}$ be the eigenvalues w/ the associated eigenfunctions

then $\begin{cases} L(\varphi) + \lambda \sigma \varphi = 0 \\ L(\bar{\varphi}) + \bar{\lambda} \sigma \bar{\varphi} = 0 \end{cases} \Rightarrow \begin{cases} \bar{\varphi} L(\varphi) + \lambda \sigma \varphi \bar{\varphi} = 0 \\ \varphi L(\bar{\varphi}) + \bar{\lambda} \sigma \varphi \bar{\varphi} = 0 \end{cases}$

$\Rightarrow \int_a^b \left(\bar{\varphi} L(\varphi) - \varphi L(\bar{\varphi})\right) dx = (\bar{\lambda} - \lambda) \int_a^b |\varphi|^2 \sigma \, dx$ but by Lagrange's identity

$\int_a^b \left(\bar{\varphi} L(\varphi) - \varphi L(\bar{\varphi})\right) dx = p\left(\bar{\varphi} \frac{d\varphi}{dx} - \varphi \frac{d\bar{\varphi}}{dx}\right)\Big|_a^b$

$= p\left(\bar{\varphi}(b) \frac{d\varphi}{dx}(b) - \varphi(b) \frac{d\bar{\varphi}}{dx}(b)\right) - p\left(\bar{\varphi}(a) \frac{d\varphi}{dx}(a) - \varphi(a) \frac{d\bar{\varphi}}{dx}(a)\right) = 0$

since $\varphi(b) = \varphi(a) = \bar{\varphi}(b) = \bar{\varphi}(a) = 0$ since the both solve the S-L prob.

so $(\bar{\lambda} - \lambda) \int_a^b |\varphi|^2 \sigma \, dx = 0$ but $|\varphi|^2 > 0$ so $\int_a^b |\varphi|^2 \sigma \, dx > 0 \Rightarrow \bar{\lambda} - \lambda = 0$

$\Rightarrow \lambda = \bar{\lambda}$

5.) $\begin{cases} u_t = \kappa u_{xx} + Q(x,t) \\ u(0,t) = A(t), \quad u(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$

consider the associated homogeneous prob. $\begin{cases} \frac{d^2h}{dx^2} + \lambda h = 0 \\ h(0) = h(L) = 0 \end{cases} \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

$\frac{d\varphi_n}{dx} = \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)$

s'pose $u(x,t) = \sum_{n=0}^{\infty} a_n(t)\,\varphi_n(x)$ so that $a_n(t) = \dfrac{\int_0^L u\,\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx}$

and $Q(x,t) = \sum_{n=0}^{\infty} q_n(t)\,\varphi_n(x)$ and $q_n(t) = \dfrac{\int_0^L Q\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx}$

then $u_t = \sum_{n=0}^{\infty} \frac{da_n}{dt}\varphi_n(x) = \kappa u_{xx} + Q(x,t) \Rightarrow \dfrac{da_n}{dt} = \kappa\dfrac{\int_0^L u_{xx}\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} + \dfrac{\int_0^L Q\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx}$

$\Rightarrow \dfrac{da_n}{dt} = \kappa\dfrac{\int_0^L u_{xx}\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} + q_n(t)$. study $\int_0^L u_{xx}\varphi_n\,dx$: using Lagrange's identity

$$\int_0^L u_{xx}\varphi_n\,dx = \int_0^L u\frac{d^2\varphi_n}{dx^2}\,dx + \left(\varphi_n u_x - u\frac{d\varphi_n}{dx}\right)\Big|_0^L = -\lambda_n\int_0^L u\varphi_n\,dx + \left(\varphi_n u_x - u\frac{d\varphi_n}{dx}\right)\Big|_0^L$$

$$= -\lambda_n\int_0^L u\varphi_n\,dx + \left(0 - B(t)\frac{n\pi}{L}(-1)^n\right) - \left(0 - \frac{n\pi}{L}A(t)\right) = -\lambda_n\int_0^L u\varphi_n\,dx + \frac{n\pi}{L}A(t) - \frac{n\pi}{L}(-1)^n B(t)$$

$$\Rightarrow \frac{da_n}{dt} = -\lambda_n\kappa\frac{\int_0^L u\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} + q_n(t) + \frac{2\kappa}{L}\left(\frac{n\pi}{L}A(t) - \frac{n\pi}{L}(-1)^n B(t)\right) = -\lambda_n\kappa a_n + K(t)$$

So $\dfrac{da_n}{dt} + \lambda_n\kappa a_n \equiv K(t) \Rightarrow a_n(t) = a_n(0)e^{-\lambda_n\kappa t} + e^{-\lambda_n\kappa t}\int_0^t K(\tau)e^{\lambda_n\kappa\tau}\,d\tau$

w/ $a_n(0) = \dfrac{\int_0^L f(x)\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} = \dfrac{2}{L}\int_0^L f(x)\sin\left(\dfrac{n\pi x}{L}\right)dx$

6.) $\begin{cases} u_t = \kappa u_{xx} + Q(x,t) \\ u_x(0,t) = A(t) \quad , u_x(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$

consider the associated homogeneous prob. $\begin{cases} \frac{d^2\phi}{dx^2} + \lambda\phi = 0 \\ \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0 \end{cases}$ $\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$

$\frac{d\varphi_n}{dx} = -\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)$

S'pose $u(x,t) = \sum_{n=0}^{\infty} a_n(t)\varphi_n(x)$ and $Q(x,t) = \sum_{n=0}^{\infty} q_n(t)\varphi_n(x)$ so that $a_n(t) = \frac{\int_0^L u\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx}$

and $q_n(t) = \frac{\int_0^L Q\varphi_n\,dx}{\int_0^L \varphi_n\,dx}$. Then $u_t = \sum_{n=0}^{\infty} \frac{da_n}{dt}\varphi_n(x) = \kappa u_{xx} + Q$

$\Rightarrow \frac{da_n}{dt} = \kappa\frac{\int_0^L u_{xx}\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} + \frac{\int_0^L Q\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} = \kappa\frac{\int_0^L u_{xx}\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} + q_n(t)$

Study $\int_0^L u_{xx}\varphi_n\,dx$: Using Lagrange's identity we get

$\int_0^L u_{xx}\varphi_n\,dx = \int_0^L u\frac{d^2\varphi_n}{dx^2}\,dx + \left(\varphi_n u_x - u\frac{d\varphi_n}{dx}\right)\Big|_0^L = -\lambda_n\int_0^L u\varphi_n\,dx + \left(\varphi_n u_x - u\frac{d\varphi_n}{dx}\right)\Big|_0^L$

$= -\lambda_n\int_0^L u\varphi_n\,dx + \left((-1)^n B(t) - 0\right) + \left(A(t) - 0\right) = -\lambda_n\int_0^L u\varphi_n\,dx + \left(A(t) + (-1)^n B(t)\right)$

$\Rightarrow \frac{da_n}{dt} = -\lambda_n\kappa\frac{\int_0^L u\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} + q_n(t) + \frac{2\kappa}{L}\left(A(t) + (-1)^n B(t)\right) = -\lambda_n\kappa a_n + K(t)$

so $\frac{da_n}{dt} + \lambda_n\kappa a_n = K(t) \Rightarrow a_n(t) = a_n(0)e^{-\lambda_n\kappa t} + e^{-\lambda_n\kappa t}\int_0^t K(\tau)e^{\lambda_n\kappa\tau}\,d\tau$

w/ $a_n(0) = \frac{\int_0^L f(x)\varphi_n(x)\,dx}{\int_0^L \varphi_n^2\,dx} = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

7.) $\begin{cases} u_t = \kappa u_{xx} + c u_x \\ u(x,0) = f(x) \quad , x \in \mathbb{R} \end{cases}$

let $U = U(\omega, t) = \hat{u}$ then taking Fourier transform of the eq. gives

$U_t = \kappa(-i\omega)^2 U + c(-i\omega) U = -(\kappa\omega^2 + ci\omega) U \Rightarrow U(\omega,t) = c(\omega) e^{-\kappa\omega^2 t} e^{-i\omega c t}$

w/ $c(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x) e^{i\omega x} dx = \hat{f}(x) = \widehat{u(x,0)}$ then $u(x,t) = \check{c}(\omega) * \left(e^{-\kappa\omega^2 t} e^{-i\omega c t}\right)^{\vee}$

by shift theorem (table) $\left(e^{-\kappa\omega^2 t} e^{-i\omega c t}\right)^{\vee} = \sqrt{\frac{\pi}{t\kappa}}\, e^{\frac{(x+ct)^2}{4t\kappa}}$

then $u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{\mathbb{R}} f(y)\, e^{\frac{(x-y+ct)^2}{4\kappa t}} dy$

8.) $\begin{cases} u_t = \kappa u_{xx} - c u \\ u(x,0) = f(x), \quad x \in \mathbb{R} \end{cases}$

let $U = U(\omega,t) = \hat{u}$ then taking the Fourier transform of the eq gives

$U_t = \kappa(-i\omega)^2 U - cU \Rightarrow U_t = -(\kappa\omega^2 + c) U \Rightarrow U(\omega,t) = c(\omega) e^{-\kappa\omega^2 t} e^{-ct}$

w/ $c(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x) e^{i\omega x} dx = \hat{f}(x) = \widehat{u(x,0)}$ then $u(x,t) = \check{c}(\omega) * \left(e^{-\kappa\omega^2 t}\right)^{\vee}$ e^{-ct} ← un-affected by F.T.

so $u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-ct} \int_{\mathbb{R}} f(y)\, e^{-\frac{(x-y)^2}{4\kappa t}} dy$

9.) $\begin{cases} u_t = \kappa u_{xx} + g(x,t) \\ u(x,0) = f(x), \quad x \in \mathbb{R} \end{cases}$

let $U = U(\omega,t) = \hat{u}$, and $Q = Q(\omega,t) = \hat{g}$ then taking Fourier transform of the eq.

$\Rightarrow U_t = \kappa(-i\omega)^2 U + Q \Rightarrow U_t = -\kappa\omega^2 U + Q$ 1st order int

so $U_t + \kappa\omega^2 U = Q \Rightarrow U(\omega,t) = c(\omega) e^{-\kappa\omega^2 t} + e^{-\kappa\omega^2 t}\int_0^t Q(\omega,\tau) e^{\kappa\omega^2\tau} d\tau$

w/ $c(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x) e^{i\omega x} dx = \hat{f}(x) = \widehat{u(x,0)}$

notice $U(\omega,t) = c(\omega) e^{-\kappa\omega^2 t} + \int_0^t Q(\omega,\tau) e^{-\kappa\omega^2(t-\tau)} d\tau$

so $u(x,t) = \check{c}(\omega) * \check{e}^{-\kappa\omega^2 t} + \int_0^t \check{Q}(\omega,\tau) \, e^{-\kappa\omega^2(t-\tau)} d\tau$

$= \frac{1}{\sqrt{4\pi\kappa t}}\int_{\mathbb{R}} f(y) e^{-\frac{(x-y)^2}{4\kappa t}} dy + \int_0^t \check{Q}(\omega,\tau) * \check{e}^{-\kappa\omega^2(t-\tau)} d\tau$

$= \frac{1}{\sqrt{4\pi\kappa t}}\int_{\mathbb{R}} f(y) e^{-\frac{(x-y)^2}{4\kappa t}} dy + \int_0^t \frac{1}{\sqrt{4\pi\kappa(t-\tau)}}\int_{\mathbb{R}} g(x-y,t) \, e^{-\frac{(x-y)^2}{4\kappa(t-\tau)}} d\tau$

10.) $\begin{cases} u_t = \kappa u_{xxx} \\ u(x,0) = f(x), \quad x \in \mathbb{R} \end{cases}$

let $U = U(\omega,t) = \hat{u}$ then taking Fourier transform of the eq gives $U_t = \kappa i\omega^3 U$

$\Rightarrow U(\omega,t) = c(\omega) e^{\kappa i\omega^3 t} = c(\omega) e^{-3\kappa t\left(\frac{-i\omega^3}{3}\right)}$ w/ $c(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x) e^{i\omega x} dx$

so $u(x,t) = \check{c}(\omega) * \check{e}^{-3\kappa t\left(\frac{-i\omega^3}{3}\right)}$ but $\check{e}^{-3\kappa t\left(-\frac{i\omega^3}{3}\right)} = \int_{\mathbb{R}} e^{-3\kappa t\left(\frac{-i\omega^3}{3}\right)} e^{-i\omega x} dx$

$= 2\pi \, Ai(-3\kappa t x)$

so $u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} f(y) \, 2\pi \, Ai(-3\kappa t(x-y)) \, dy$

$= \int_{\mathbb{R}} f(y) \, Ai(-3\kappa t(x-y)) \, dy$

11.) $\begin{cases} \dfrac{d^2y}{dx^2} + (\lambda - V(x))y = 0 \\ y(x) = 0, \quad y \to 0 \text{ as } x \to \pm\infty \end{cases}$

let $-\varkappa_n^2$ be eigenvalue in discrete spectrum $\longleftrightarrow \varphi_n(x)$ and let μ^2 be eigenvalue from the cont. spectrum $\longleftrightarrow \nu(x)$

then $\begin{cases} \dfrac{d^2\varphi_n}{dx^2} - (\varkappa_n^2 + V(x))\varphi_n = 0 \\ \dfrac{d^2\nu}{dx^2} + (\mu^2 - V(x))\nu = 0 \end{cases} \Rightarrow \begin{cases} \nu\varphi_n'' - (\varkappa_n^2 + V(x))\varphi_n\nu = 0 \\ \varphi_n\nu'' + (\mu^2 - V(x))\varphi_n\nu = 0 \end{cases}$

$\Rightarrow \nu\varphi_n'' - \varphi_n\nu'' = (\varkappa_n^2 + \mu^2)\varphi_n\nu \Rightarrow \int_{\mathbb{R}}(\nu\varphi_n'' - \varphi_n\nu'')dx = (\varkappa_n^2 + \mu^2)\int_{\mathbb{R}}\varphi_n\nu\, dx$

using Lagrange identity $\Rightarrow \int_{\mathbb{R}}(\nu\varphi_n'' - \varphi_n\nu'')dx = (\nu\varphi_n' - \varphi_n\nu')\Big|_{-\infty}^{\infty}$

but since ν, φ_n are solns to the eq. $\nu, \varphi_n \to 0$ as $x \to \pm\infty$ and ν', φ_n' are bnded

so $\int_{\mathbb{R}}(\nu\varphi_n'' - \varphi_n\nu'')dx = 0 \Rightarrow (\varkappa_n^2 + \mu^2)\int_{\mathbb{R}}\varphi_n\nu\, dx = 0$ but $\varkappa_n^2 + \mu^2 \neq 0$

so $\int_{\mathbb{R}}\varphi_n\nu\, dx = 0$.

12.) $\begin{cases} \dfrac{d^2y}{dx^2} + (\lambda - V(x))y = 0 \\ y(x) = 0, \; y \to 0 \text{ as } x \to \pm\infty \end{cases}$

let $-\varkappa_n^2, -\varkappa_m^2$ be different eigenvalues $\longleftrightarrow \varphi_n(x), \varphi_m(x)$ eigen fncs.

then $\begin{cases} \varphi_n'' - (\varkappa_n^2 + V(x))\varphi_n = 0 \\ \varphi_m'' - (\varkappa_m^2 + V(x))\varphi_m = 0 \end{cases} \Rightarrow \begin{cases} \varphi_m\varphi_n'' - (\varkappa_n^2 + V(x))\varphi_n\varphi_m = 0 \\ \varphi_n\varphi_m'' - (\varkappa_m^2 + V(x))\varphi_n\varphi_m = 0 \end{cases}$

$\Rightarrow \int_{\mathbb{R}}(\varphi_m\varphi_n'' - \varphi_n\varphi_m'')dx = \int_{\mathbb{R}}(\varkappa_m^2 - \varkappa_n^2)\varphi_n\varphi_m\, dx = (\varkappa_m^2 - \varkappa_n^2)\int_{\mathbb{R}}\varphi_n\varphi_m\, dx$

using Lagrange id. $\Rightarrow \int_{\mathbb{R}}(\varphi_m\varphi_n'' - \varphi_n\varphi_m'')dx = (\varphi_m\varphi_n' - \varphi_n\varphi_m')\Big|_{-\infty}^{\infty}$

since φ_n, φ_m are solns to the eq. $\varphi_n, \varphi_m \to 0$ as $x \to \pm\infty$ and φ_n', φ_m' bnded

so $\int_{\mathbb{R}}(\varphi_m\varphi_n'' - \varphi_n\varphi_m'')dx = 0 \Rightarrow (\varkappa_m^2 - \varkappa_n^2)\int_{\mathbb{R}}\varphi_n\varphi_m\, dx = 0$ but $\varkappa_m^2 \neq \varkappa_n^2$

so $\int_{\mathbb{R}}\varphi_n\varphi_m\, dx = 0$

13.) let $u(\vec{x},t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(\vec{y}) e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} d\vec{y} = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(\vec{y}) e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} d\vec{y}$

so $u_t = \underbrace{-\frac{n}{2}(4\pi t)^{-\frac{n}{2}-1}(4\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(\vec{y}) e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} d\vec{y}}_{a} + \underbrace{(4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(\vec{y}) e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} \frac{\|\vec{x}-\vec{y}\|^2}{4t^2} d\vec{y}}_{b}$

then $\Delta u = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(\vec{y}) \, \Delta \left(e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} \right) d\vec{y}$

need to study $\Delta \left(e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} \right)$, but $\frac{\partial}{\partial x_i} \left(e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} \right) = e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} \frac{\partial}{\partial x_i} \left(\frac{-\|\vec{x}-\vec{y}\|^2}{4t} \right)$

but $\frac{\partial}{\partial x_i} (\|\vec{x}\|^2) = \frac{\partial}{\partial x_i} (x_1^2 + x_2^2 + \dots + x_n^2) = 2x_i$ so $\frac{\partial}{\partial x_i} \left(e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} \right) = \frac{-(x_i - y_i)}{2t} e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}}$

then $\frac{\partial^2}{\partial x_i^2} \left(e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} \right) = \frac{\partial}{\partial x_i} \left(\frac{-(x_i-y_i)}{2t} e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} \right) = \frac{\partial}{\partial x_i} \left(\frac{-(x_i-y_i)}{2t} \right) e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} + \frac{-(x_i-y_i)}{2t} \frac{\partial}{\partial x_i} \left(e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} \right)$

$= \frac{-1}{2t} e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} + \frac{(x_i-y_i)^2}{4t^2} e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}}$

so $\Delta \left(e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} \right) = \left(\sum_{i=1}^{n} \frac{-1}{2t} + \sum_{i=1}^{n} \frac{(x_i-y_i)^2}{4t^2} \right) e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} = \left(-\frac{n}{2t} + \frac{\|\vec{x}-\vec{y}\|^2}{4t^2} \right) e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}}$

so $\Delta u = \underbrace{-\frac{n}{2t} (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(\vec{y}) e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}} d\vec{y}}_{c} + \underbrace{(4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(\vec{y}) \frac{\|\vec{x}-\vec{y}\|^2}{4t^2} e^{-\frac{\|\vec{x}-\vec{y}\|^2}{4t}}}_{d} d\vec{y}$

we see now $a = c$ and $b = d$ thus $u_t = \Delta u$.