1.) $\begin{cases} \Delta v + \lambda v = 0 \\ v|_{\partial\Omega} = 0 \end{cases}$

let $\lambda_1 \xrightarrow{\text{associated}}$ to φ_1 and $\lambda_2 \xrightarrow{\text{associated}}$ to φ_2 for $\lambda_1 \neq \lambda_2$ then they both solve

$\begin{cases} \Delta\varphi_1 = -\lambda_1\varphi_1 \\ \varphi_1|_{\partial\Omega} = 0 \end{cases}$ and $\begin{cases} \Delta\varphi_2 = -\lambda_2\varphi_2 \\ \varphi_2|_{\partial\Omega} = 0 \end{cases}$ $\Rightarrow \begin{cases} \varphi_2\Delta\varphi_1 = -\lambda_1\varphi_1\varphi_2 \\ \varphi_1\Delta\varphi_2 = -\lambda_2\varphi_1\varphi_2 \end{cases}$

$\Rightarrow (\lambda_1 - \lambda_2)\varphi_1\varphi_2 = \varphi_1\Delta\varphi_2 - \varphi_2\Delta\varphi_1 \Rightarrow (\lambda_1-\lambda_2)\iint_\Omega \varphi_1\varphi_2\, dA = \iint_\Omega (\varphi_1\Delta\varphi_2 - \varphi_2\Delta\varphi_1)\, dA$

$= \int_{\partial\Omega} (\varphi_1\nabla\varphi_2 - \varphi_2\nabla\varphi_1)\cdot\vec{n}\, ds$ by Green's Formula, but $\varphi_1|_{\partial\Omega} = 0$

$\Rightarrow \int_{\partial\Omega} \varphi_1\nabla\varphi_2 \cdot \vec{n}\, ds = 0$ and similarly $\varphi_2|_{\partial\Omega} = 0$ so $\int_{\partial\Omega}\varphi_2\nabla\varphi_1\cdot\vec{n}\, ds = 0$

$\Rightarrow (\lambda_1 - \lambda_2)\iint_\Omega \varphi_1\varphi_2\, dA = 0$ since $\lambda_1 \neq \lambda_2 \Rightarrow \iint_\Omega \varphi_1\varphi_2\, dA = 0$ so φ_1, φ_2 are orthogonal

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2.) $\begin{cases} \Delta v + \lambda v = 0 \\ v|_{\partial\Omega} = 0 \end{cases}$

let $\lambda \xrightarrow{\text{assoc.}}$ to φ. Then by conjugation know $\bar\lambda \to \bar\varphi$ and they both solve

$\begin{cases} \Delta\varphi = -\lambda\varphi \\ \varphi|_{\partial\Omega} = 0 \end{cases}$ and $\begin{cases} \Delta\bar\varphi = -\bar\lambda\bar\varphi \\ \bar\varphi|_{\partial\Omega} = 0 \end{cases}$ $\Rightarrow \begin{cases} \bar\varphi\Delta\varphi = -\lambda\varphi\bar\varphi \\ \varphi\Delta\bar\varphi = -\bar\lambda\bar\varphi\varphi \end{cases}$

$\Rightarrow (\lambda - \bar\lambda)\varphi\bar\varphi = \varphi\Delta\bar\varphi - \bar\varphi\Delta\varphi \Rightarrow (\lambda - \bar\lambda)\iint_\Omega |\varphi|^2\, dA = \iint_\Omega (\varphi\Delta\bar\varphi - \bar\varphi\Delta\varphi)\, dA$

$= \int_{\partial\Omega} (\varphi\nabla\bar\varphi - \bar\varphi\nabla\varphi)\cdot\vec{n}\, ds$ by Green's Formula. but $\varphi|_{\partial\Omega} = 0$

$\Rightarrow \int_{\partial\Omega} \varphi\nabla\bar\varphi\cdot\vec{n}\, ds = 0$ and $\bar\varphi|_{\partial\Omega} = 0 \Rightarrow \int_{\partial\Omega} \bar\varphi\nabla\varphi\cdot\vec{n}\, ds = 0$

so $(\lambda - \bar\lambda)\iint_\Omega |\varphi|^2\, dA = 0$ but the integral is pos. $\Rightarrow \lambda - \bar\lambda = 0 \Rightarrow \lambda = \bar\lambda$ so λ real.

3) $(x^6+x^2)\frac{d^2y}{dx^2}+(x^7+x)\frac{dy}{dx}+(6x^5-4)y=0$

want to guess soln's of form $y=x^p$ near zero so can throw away x^6 in $\frac{d^2y}{dx^2}$

x^7 in $\frac{dy}{dx}$ and $6x^5$ in y terms otherwise powers won't match.

$\Rightarrow$ $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y\approx 0$ so $y=x^p$ and $y'=px^{p-1}$ and $y''=p(p-1)x^{p-2}$

plugging in gives $p(p-1)x^p+px^p-4x^p\approx 0$ $\Rightarrow$ $p(p-1)+p-4=0$

$p^2-p+p-4=0$ so $p^2-4=0$ so $p=\pm 2$

so $y\approx c_1x^2+c_2x^{-2}+\ldots$ $\leftarrow$ lower terms we don't know.

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4) $(x^6+x^2)\frac{d^2y}{dx^2}+\left(x^5+\frac{3}{16}\right)y=0$

want to guess soln's of form $y=x^p$ near zero so can throw away

x^6 in y'' and x^5 in y terms otherwise powers won't match

$\Rightarrow$ $x^2\frac{d^2y}{dx^2}+\frac{3}{16}y\approx 0$ so $y=x^p$ and $y'=px^{p-1}$ and $y''=p(p-1)x^{p-2}$

plugging in gives $p(p-1)x^p+\frac{3}{16}x^p\approx 0$ $\Rightarrow$ $p(p-1)+\frac{3}{16}=0$

$\Rightarrow$ $p^2-p+\frac{3}{16}=0$ or $16p^2-16p+3=0$ so $p=\frac{16\pm\sqrt{256-192}}{32}=\frac{1}{2}\pm\frac{8}{32}$

$=\frac{1}{2}\pm\frac{1}{4}$

so $p=\frac{3}{4},\frac{1}{4}$ so $y\approx c_1x^{\frac{3}{4}}+c_2x^{\frac{1}{4}}+\ldots$ $\leftarrow$ lower terms we don't know

50.) $\begin{cases} u_t = \kappa u_{xx} + Q(x,t) \\ u_x(0,t) = A(t), \quad u(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$

The associated homogeneous problem is $\begin{cases} \dfrac{d^2 h}{dx^2} + \lambda h = 0 \\ \dfrac{dh}{dx}(0) = h(L) = 0 \end{cases}$ know $\lambda_n = \left(\dfrac{(2n-1)\pi}{2L}\right)^2$ $n = 1, 2, \ldots$

and sol'n's $\varphi_n(x) = \cos\left(\dfrac{(2n-1)\pi x}{2L}\right)$

$\dfrac{d\varphi_n}{dx} = \dfrac{-(2n-1)\pi}{2L} \sin\left(\dfrac{(2n-1)\pi x}{2L}\right)$

s'pose $u(x,t) = \sum_{n=0}^{\infty} b_n(t)\varphi_n(x)$ then $b_n(t) = \dfrac{\int_0^L u\,\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx}$

and s'pose $Q(x,t) = \sum_{n=0}^{\infty} q_n(t)\varphi_n(x)$ then $q_n(t) = \dfrac{\int_0^L Q\,\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx}$

then $u_t = \sum_{n=0}^{\infty} \dfrac{db_n}{dt}\varphi_n(x) = \kappa u_{xx} + Q$

so $\dfrac{db_n}{dt} = \dfrac{\kappa\int_0^L u_{xx}\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx} + \dfrac{\int_0^L Q\,\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx} = \kappa\dfrac{\int_0^L u_{xx}\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx} + q_n(t)$

Using Green's formula $\int_0^L u_{xx}\varphi_n(x)\,dx = \int_0^L u\dfrac{d^2\varphi_n}{dx^2}\,dx + \left(\varphi_n u_x - u\dfrac{d\varphi_n}{dx}\right)\Big|_0^L$

$= -\lambda_n \int_0^L u\,\varphi_n\,dx + \left(\varphi_n(L)u_x(L) - u(L)\dfrac{d\varphi_n}{dx}(L)\right) - \left(\varphi_n(0)u_x(0) - u(0)\dfrac{d\varphi_n}{dx}(0)\right)$

$= -\lambda_n \int_0^L u\,\varphi_n\,dx + 0 - B(t)(-1)^n - (A(t) - 0) = -\lambda_n \int_0^L u\,\varphi_n\,dx - (A(t) + (-1)^n B(t))$

so $\dfrac{db_n}{dt} = -\lambda \kappa b_n + q_n(t) - (A(t) + (-1)^n B(t)) \Rightarrow \dfrac{db_n}{dt} + \lambda_n \kappa b_n = q_n(t) - (A(t) + (-1)^n B(t))$

is a 1st order eq. and integrating factor is $\mu = \exp\left(\int \lambda_n \kappa\,dt\right) = e^{\lambda_n \kappa t}$

$\Rightarrow \left(b_n e^{\lambda_n \kappa t}\right)' = \left[q_n(t) - (A(t) + (-1)^n B(t))\right]e^{\lambda_n \kappa t}$

$\Rightarrow b_n e^{\lambda_n \kappa t} = b_n(0) + \int_0^t \left[q_n(\tau) - (A(\tau) + (-1)^n B(\tau))\right]e^{\lambda_n \kappa \tau}\,d\tau$

So $b_n(t) = b_n(0)e^{-\lambda_n \kappa t} + e^{-\lambda_n \kappa t}\int_0^t \left[q_n(\tau) - (A(\tau) + (-1)^n B(\tau))\right]e^{\lambda_n \kappa \tau}\,d\tau$

so $u(x,t) = \sum_{n=0}^{\infty} b_n(t)\cos\left(\dfrac{(2n-1)\pi x}{2L}\right)$ w/ $b_n(t)$ and $b_n(0) = \dfrac{2}{L}\int_0^L f(x)\cos\left(\dfrac{(2n-1)\pi x}{2L}\right)dx$

(6.) $\begin{cases} u_t = \kappa u_{xx} + Q(x,t) \\ u_x(0,t) = A(t), \quad u_x(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$

the associated homogeneous prob is $\begin{cases} \frac{d^2 h}{dx^2} + \lambda h = 0 \\ \frac{dh}{dx}(0) = \frac{dh}{dx}(L) = 0 \end{cases}$ known $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, \ldots$ and soln's $\varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$

S'pose $u(x,t) = \sum_{n=0}^{\infty} b_n(t)\varphi_n(x)$ then $b_n(t) = \frac{\int_0^L u\varphi_n(x)dx}{\int_0^L \varphi_n^2(x)dx}$

and s'pose $Q(x,t) = \sum_{n=0}^{\infty} q_n(t)\varphi_n(x)$ then $q_n(t) = \frac{\int_0^L Q\varphi_n(x)dx}{\int_0^L \varphi_n^2(x)dx}$

Then $u_t = \sum_{n=0}^{\infty} \frac{db_n}{dt}\varphi_n(x) = \kappa u_{xx} + Q$

so $\frac{db_n}{dt} = \kappa \frac{\int_0^L u_{xx}\varphi_n dx}{\int_0^L \varphi_n^2 dx} + \frac{\int_0^L Q\varphi_n dx}{\int_0^L \varphi_n^2 dx} = \kappa \frac{\int_0^L u_{xx}\varphi_n dx}{\int_0^L \varphi_n^2 dx} + q_n(t)$

using Green's formula $\int_0^L u_{xx}\varphi_n dx = \int_0^L u\frac{d^2\varphi_n}{dx^2}dx + \left(\varphi_n u_x - u\frac{d\varphi_n}{dx}\right)\Big|_0^L$

$= -\lambda_n\int_0^L u\varphi_n dx + \left(\varphi_n(L)u_x(L) - u(L)\frac{d\varphi_n}{dx}(L)\right) - \left(\varphi_n(0)u_x(0) - u(0)\frac{d\varphi_n}{dx}(0)\right)$

$= -\lambda_n\int_0^L u\varphi_n dx + (-1)^n B(t) - A(t) \Rightarrow \frac{db_n}{dt} = -\lambda_n\kappa b_n + (-1)^n B(t) - A(t) + q_n(t)$

so $\frac{db_n}{dt} + \lambda_n\kappa b_n = q_n(t) + (-1)^n B(t) - A(t)$ is a 1st order eq w/ integrating factor

$\mu = \exp\left(\int \lambda_n\kappa\, dt\right) = e^{\lambda_n\kappa t} \Rightarrow \left(b_n e^{\lambda_n\kappa t}\right)' = \left[q_n(t) + (-1)^n B(t) - A(t)\right]e^{\lambda_n\kappa t}$

$\Rightarrow b_n e^{\lambda_n\kappa t} = b_n(0) + \int_0^t \left[q_n(\tau) + (-1)^n A(\tau) - B(\tau)\right]e^{\lambda_n\kappa\tau}d\tau$

so $b_n(t) = b_n(0)e^{-\lambda_n\kappa t} + e^{-\lambda_n\kappa t}\int_0^t \left[q_n(\tau) + (-1)^n A(\tau) - B(\tau)\right]e^{\lambda_n\kappa\tau}d\tau$

so $u(x,t) = \sum_{n=0}^{\infty} b_n(t)\cos\left(\frac{n\pi x}{L}\right)$ w/ $b_n(t)$ and $b_n(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

7.) $\begin{cases} u_t = \kappa u_{xx} + g(x,t) \\ u(x,0) = f(x), \quad x \in \mathbb{R} \end{cases}$

let $U = U(\omega,t) = \hat{u}$, and $Q = Q(\omega,t) = \hat{g}$ then taking Fourier transform of the eq.

$\Rightarrow \quad U_t = \kappa(-i\omega)^2 U + Q \quad \Rightarrow \quad U_t = -\kappa\omega^2 U + Q \quad$ 1st order int

so $\quad U_t + \kappa\omega^2 U = Q \quad \Rightarrow \quad U(\omega,t) = c(\omega)e^{-\kappa\omega^2 t} + e^{-\kappa\omega^2 t}\int_0^t Q(\omega,\tau)e^{\kappa\omega^2\tau}\,d\tau$

w/ $c(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x)e^{i\omega x}\,dx = \widehat{f(x)} = \widehat{u(x,0)}$

notice $\quad U(\omega,t) = c(\omega)e^{-\kappa\omega^2 t} + \int_0^t Q(\omega,\tau)\,e^{-\kappa\omega^2(t-\tau)}\,d\tau$

so $\quad u(x,t) = \check{c}(\omega) * \check{e}^{-\kappa\omega^2 t} + \int_0^t Q(\omega,\tau)\,e^{-\kappa\omega^2(t-\tau)}\,d\tau$

$= \frac{1}{\sqrt{4\pi\kappa t}}\int_{\mathbb{R}} f(y)\,e^{-\frac{(x-y)^2}{4\kappa t}}\,dy + \int_0^t \check{Q}(\omega,\tau) * \check{e}^{-\kappa\omega^2(t-\tau)}\,d\tau$

$= \frac{1}{\sqrt{4\pi\kappa t}}\int_{\mathbb{R}} f(y)\,e^{-\frac{(x-y)^2}{4\kappa t}}\,dy + \int_0^t \frac{1}{\sqrt{4\pi\kappa(t-\tau)}}\int_{\mathbb{R}} g(x-y,t)\,e^{-\frac{(x-y)^2}{4\kappa(t-\tau)}}\,d\tau$

8.) $\begin{cases} u_t = \kappa u_{xxx} \\ u(x,0) = f(x), \quad x \in \mathbb{R} \end{cases}$

let $U = U(\omega,t) = \hat{u}$ then taking Fourier transform of the eq gives $U_t = \kappa i\omega^3 U$

$\Rightarrow \quad U(\omega,t) = c(\omega)e^{\kappa i\omega^3 t} = c(\omega)e^{-3\kappa t\left(\frac{-i\omega^3}{3}\right)} \qquad$ w/ $c(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x)e^{i\omega x}\,dx$

so $\quad u(x,t) = \check{c}(\omega) * \check{e}^{-3\kappa t\left(\frac{-i\omega^3}{3}\right)} \qquad$ but $\check{e}^{-3\kappa t\left(-\frac{i\omega^3}{3}\right)} = \int_{\mathbb{R}} e^{-3\kappa t\left(\frac{-i\omega^3}{3}\right)}e^{-i\omega x}\,dx$

$= 2\pi\, Ai(-3\kappa t x)$

so $\quad u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} f(y)\, 2\pi\, Ai(-3\kappa t(x-y))\,dy$

$= \int_{\mathbb{R}} f(y)\, Ai(-3\kappa t(x-y))\,dy$

9.) ~~want~~ $u(x,y,t) = g(x-b_1 t,\ y-b_2 t) + \int_0^t f(x+(s-t)b_1,\ y+(s-t)b_2,\ s)\,ds$

set $\alpha = x - b_1 t$ $\qquad \gamma = x+(s-t)b_1$

$\beta = y - b_2 t$ $\qquad \delta = y+(s-t)b_2$

then $u(x,y,t) = g(\alpha, \beta) + \int_0^t f(\gamma, \delta, s)\,ds$

so $u_t = \frac{\partial g}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial g}{\partial \beta}\frac{\partial \beta}{dt} + f(x,y,t) + \int_0^t \left(\frac{\partial f}{\partial \gamma}\frac{\partial \gamma}{\partial t} + \frac{\partial f}{\partial \delta}\frac{\partial \delta}{\partial t}\right) ds$

$= -\frac{\partial g}{\partial \alpha} b_1 - \frac{\partial g}{\partial \beta} b_2 + f(x,y,t) - \int_0^t \left(\frac{\partial f}{\partial \gamma} b_1 + \frac{\partial f}{\partial \delta} b_2\right) ds$

then $\nabla u = \left(\frac{\partial g}{\partial \alpha}\frac{\partial \alpha}{\partial x},\ \frac{\partial g}{\partial \beta}\frac{\partial \beta}{\partial x}\right) + \int_0^t \left(\frac{\partial f}{\partial \gamma}\frac{\partial \gamma}{\partial x},\ \frac{\partial f}{\partial \delta}\frac{\partial \delta}{\partial y}\right) ds$

$= \left(\frac{\partial g}{\partial \alpha},\ \frac{\partial g}{\partial \beta}\right) + \int_0^t \left(\frac{\partial f}{\partial \gamma},\ \frac{\partial f}{\partial \delta}\right) ds$

so $\vec{b}\cdot\nabla u = b_1 \frac{\partial g}{\partial \alpha} + b_2 \frac{\partial g}{\partial \beta} + \int_0^t \left(b_1 \frac{\partial f}{\partial \gamma} + b_2 \frac{\partial f}{\partial \delta}\right) ds$

so $u_t + \vec{b}\cdot\nabla u = -\frac{\partial g}{\partial \alpha} b_1 - \frac{\partial g}{\partial \beta} b_2 - \int_0^t \left(\frac{\partial f}{\partial \gamma} b_1 + \frac{\partial f}{\partial \delta} b_2\right) ds + f(x,y,t)$

$+ \frac{\partial g}{\partial \alpha} b_1 + \frac{\partial g}{\partial \beta} b_2 + \int_0^t \left(\frac{\partial f}{\partial \gamma} b_1 + \frac{\partial f}{\partial \delta} b_2\right)$

$= f(x,y,t)$