D-bar Operators in Commutative and Noncommutative Domains

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October 19, 2013

Introduction and Relavent Papers

- Atiyah, M. F., Patodi, V. K. and Singer I. M., Spectral asymmetry and Riemannian geometry I. *Math. Proc. Camb. Phil. Soc.*, 77, 43 - 69, 1975.

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- Carey, A. L., Klimek, S. and Wojciechowski, K. P., Dirac operators on noncommutative manifolds with boundary, *Lett. Math. Phys.* 93, 107 - 125, 2010.

Commutative Disk and Annulus

Spaces and Operator

- Let $\mathbb{D}_{w^+} = \{z \in \mathbb{C} \ : \ |z| \leq w^+\}$ and $\mathbb{A}_{w^-,w^+} = \{z \in \mathbb{C} \ : \ 0 < w^- \leq |z| \leq w^+\}$.

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- Notice $\partial\mathbb{D}_{w^+} \cong S^1$ and $\partial\mathbb{A}_{w^-,w^+} \cong S^1 \cup S^1$.

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- Notice $\partial\mathbb{D}_{w^+} \cong S^1$ and $\partial\mathbb{A}_{w^-,w^+} \cong S^1 \cup S^1$.
- Let

$$D = \frac{\partial}{\partial \overline{z}}$$

be the operator acting on the space of H^1 functions.

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- Let $\sigma_\pm f(\varphi) = f(w^\pm e^{i\varphi})$.
- $$0 \to C_0(\mathbb{A}_{w^-,w^+}) \to C(\mathbb{A}_{w^-,w^+}) \xrightarrow{\sigma_+ \oplus \sigma_-} C(S^1) \oplus C(S^1) \to 0$$

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- Let D have a "special" decomposition structure so that it extends naturally to the infinite cylinder.

APS Boundary Condition

Study D with domain:

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- There is a $F^{\mathrm{ext}} \in H^1_{\mathrm{loc}}(cylinder)$ such that $DF^{\mathrm{ext}} = 0$, $F^{\mathrm{ext}}\big|_Y = F|_Y$ and $F^{\mathrm{ext}} \in L^2(cylinder)$

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- and $\mathrm{dom}(D_{\mathbb{A}})$ consists of $a \in H^1(\mathbb{A}_{w^-,w^+})$ such that there is $a^{\mathrm{ext}} \in H^1_{\mathrm{loc}}(\mathbb{C} \setminus \mathbb{A}_{w^-,w^+})$ such that $a^{\mathrm{ext}}\big|_{S^1 \cup S^1} = a\big|_{S^1 \cup S^1}$, $Da^{\mathrm{ext}} = 0$, $a^{\mathrm{ext}} \in L^2(\mathbb{C} \setminus \mathbb{A}_{w^-,w^+})$.

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- $D_{\mathbb{D}}$ and $D_{\mathbb{A}}$ have parametrices, i.e. they are almost invertible.

A Fourier Series and Boundary Conditions Equivalence

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- Let $a \in \mathrm{dom}(D_{\mathbb{A}})$, then $f_n(w^+) = 0$ for $n \geq 0$ and $g_n(w^-) = 0$ for $n \geq 1$.

Dirac Operator in Polar Form and Parametrix Decomposition

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- $$Qa = -\sum_{n=0}^{\infty} e^{in\varphi} \int_r^{w^+} f_{n+1}(\rho)\frac{r^{n-1}}{\rho^n} d\rho + \sum_{n=1}^{\infty} e^{-in\varphi} \int_{w^-}^{r} g_{n-1}(\rho)\frac{\rho^{n-1}}{r^n} d\rho$$

Results

Theorem

The operators $D_{\mathbb{D}}$ and $D_{\mathbb{A}}$ are unbounded Fredholm operators. Moreover their respective parametrices $Q_{\mathbb{D}}$ and $Q_{\mathbb{A}}$ are compact operators. This also means these are elliptic boundary value problems.

Quantum Disk

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- $C(X)$ commutative C^*-algebra with unit
- GN says if $\mathcal{A}$ is a commutative C^*-algebra with unit, then there is a X, compact topological space such that $\mathcal{A} = C(X)$
- We think of a noncommutative (quantum) space as a noncommutative C^*-algebra.

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- Define $U_W e_k = w(k)e_{k+1}$ where $\{w(k)\}_{k\in\mathbb{N}}$ is an increasing sequence of positive real numbers such that

$$w^+ := \lim_{k\to\infty} w(k)$$

exists.

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- $\sigma(U) = e^{i\varphi}$, $\sigma(U^*) = e^{-i\varphi}$, $\sigma(\mathrm{compact}) = 0$
- This C^*-algebra is the quantum disk.

Informal Idea About Noncommutative Spaces

Classical

- $\mathbb{D} \longrightarrow C(\mathbb{D})$ C^*-algebra generated by z and $\overline{z}$

Quantum

- $\mathbb{D}_q \longrightarrow C^*(U_W)$, generated by unilateral shift

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Quantum

- $\mathbb{D}_q \longrightarrow C^*(U_W)$, generated by unilateral shift
- $\mathbb{A}_q \longrightarrow C^*(U_W)$, generated by bilateral shift

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- $f_n(k) = \langle e_k, (U^*)^n a e_k \rangle$, $g_n(k) = \langle e_k, aU^n e_k \rangle$

A Formal Series II

- $$\|a_{\text{series}}\|^2 = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{a^{(n)}(k)}|f_n(k)|^2 + \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{a^{(n)}(k)}|g_n(k)|^2$$

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- This series looks very similar to the Fourier series for the classical case. Think of k as the discretization of the radial variable r where we divide up the unit interval into infinitely many subintervals so the $1/a^{(n)}(k)$ appear as the differential term in the integral for the norm.

Hilbert Space

- Let $\mathcal{H}$ be the Hilbert space consisting of the formal series a_{series} such that $\|a_{\text{series}}\|$ is finite.

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- **Proposition**

  If $a \in C^*(U_W)$, then a_{series} converges to a in $\mathcal{H}$ and moreover $C^*(U_W)$ is dense in $\mathcal{H}$.

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The Operator

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- $\mathrm{dom}(D) = \{a \in \mathcal{H} \ : \ Da \in \mathcal{H}\}$

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- If $a \in \mathrm{dom}(D)$, then $f_n(\infty) = 0$ for $n \geq 0$.
- D also has a parametrix Q.

Unbounded Jacobi Operators

- $$A^{(n)}h(k) = a^{(n)}(k)\left(h(k) - c^{(n)}(k-1)h(k-1)\right)$$
  $$\overline{A}^{(n)}h(k) = a^{(n+1)}(k)\left(h(k) - c^{(n)}(k)h(k+1)\right)$$

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$$\overline{A}^{(n)}h(k) = a^{(n+1)}(k)\left(h(k) - c^{(n)}(k)h(k+1)\right)$$
- where $c^{(n)}(k) = w(k)/w(k+n+1)$

Operator Decomposition

- $$Da = -\sum_{n=0}^{\infty} U^{n+1}\overline{A}^{(n)}W^{(n)}f_n(K) + \sum_{n=1}^{\infty} W^{(n-1)}A^{(n-1)}g_n(K)(U^*)^{n-1}$$

The Parametrix

$$Qa =$$

$$- \sum_{m=0}^{\infty} U^n \left(\sum_{i=k}^{\infty} \prod_{j=1}^{n} \frac{w(k+j)}{w(i+j)} \cdot \frac{S^{1/2}(i) S^{1/2}(i+n+1)}{w(k+n)} f_{n+1}(i) \right)$$

$$+ \sum_{n=1}^{\infty} \left(\sum_{i=0}^{k} \prod_{j=0}^{n-1} \frac{w(i+j)}{w(k+j)} \cdot \frac{S^{1/2}(i) S^{1/2}(i+n-1)}{w(i+n-1)} g_{n-1}(i) \right) (U^*)^n$$

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The operator D is an unbounded Fredholm operator. Moreover it's parametrix Q is a compact operator and hence this is an elliptic boundary value problem.

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The End

Thank You