Analysis of p-Adic Numbers

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Setup

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- $\mathbb{Q}_p$: field of p-adic numbers, is the completion of the field $\mathbb{Q}$ w.r.t. the norm $|\cdot|_p$

Facts and Examples

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- $-\gamma$ is called the order of x and denoted $\mathrm{ord}\ x = -\gamma$ and $\mathrm{ord}\ 0 := -\infty$

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- Means the norm in $\mathbb{Q}_p$ is an ultrametric

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- Have the following relations:

$$B_\gamma(a) = \bigcup_{\gamma' \leq \gamma} S_{\gamma'}(a), \quad S_\gamma(a) = B_\gamma(a) \setminus B_{\gamma-1}(a)$$

$$\mathbb{Q}_p = \bigcup_{\gamma \in \mathbb{Z}} B_\gamma(a), \quad \mathbb{Q}_p^\times = \bigcup_{\gamma \in \mathbb{Z}} S_\gamma(a)$$

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- Every point in a disk is its center
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- A disk is a finite union of disjoint disks of smaller radii
- If $x \in B_\gamma(a)$, $B_{\gamma'}(b)$, then $B_\gamma(a) \subset B_{\gamma'}(b)$ or $B_{\gamma'}(b) \subset B_\gamma(a)$
- A disk is open and compact

Some Calculus of $\mathbb{Q}_p$

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- Normally have dx in regular calculus, here we have $d_p x$.
- $d_p x$ has the following properties:
- 1.) $d_p(x+a) = d_p x$ for $a \in \mathbb{Q}_p$, 2.) $d_p(ax) = |a|_p d_p x$ for $a \in \mathbb{Q}_p^{\times}$

Some Calculus of $\mathbb{Q}_p$

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- Normally have dx in regular calculus, here we have d_px.
- d_px has the following properties:
- 1.) $d_p(x+a) = d_px$ for $a \in \mathbb{Q}_p$, 2.) $d_p(ax) = |a|_p d_px$ for $a \in \mathbb{Q}_p^\times$
- We have the following integrals:

$$\int_{Z_p} d_px = 1, \quad \int_{B_\gamma} d_px = p^\gamma, \quad \int_{S_\gamma} d_px = \left(1 - \frac{1}{p}\right) p^\gamma$$

$$\int_{\mathbb{Q}_p} f(x)d_px = \sum_{\gamma=-\infty}^{\infty} \int_{S_\gamma} f(x)d_px, \quad \int_{B_\gamma} |x|_p^{\alpha-1} = \frac{1-p^{-1}}{1-p^{-\alpha}} p^{\alpha\gamma}, \ \alpha > 0$$

Differential Operators

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- The goal is to find functions u that satisfy this type of boundary value problem.

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- This has been studied very extensively classically, meaning when $L = \frac{d}{dx}$ or

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$$L = a_n(x)\frac{d^n}{dx^n} + \cdots + a_1(x)\frac{d}{dx} + a_0(x),$$

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- These types of questions will lead to problems in ODEs or PDEs. Current research still goes on in this type of question.

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- Now L becomes some kind of difference of the vertices and edges making it seem like the problem becomes a discrete problem.
- Now one has to study function spaces of sequences and series that relate to $\mathbb{Q}_p$

References

Vladimirov, V.S., Table of Integrals of Complex-valued Functions of p-Adic Arguments, arXiv:math-ph/9911027v1 22 Nov 1999.

The End

Thank You