Dirac operators on the solid torus with global boundary conditions

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Introduction and Relavent Papers

- Atiyah, M. F., Patodi, V. K. and Singer I. M., Spectral asymmetry and Riemannian geometry I. *Math. Proc. Camb. Phil. Soc.*, 77, 43 - 69, 1975.

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- Mishchenko, A. V. and Sitenko, Yu, Spectral Boundary Conditions and Index Theorem for Two-Dimensional Compact Manifold with Boundary, *Annals of Physics*, 218, 199 - 232, 1992.

Commutative Solid Torus

Setup, Hilbert Space and a Short Exact Sequence

- Let $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| \leq 1\}$, $S^1 = \{e^{i\theta} \ : \ 0 \leq \theta \leq 2\pi\}$, and $\mathbb{T}^2$ the 2-dimensional torus

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- Inner product of $\mathcal{H}$ denoted $\langle F, G \rangle$ for $F, G \in \mathcal{H}$.
- $0 \longrightarrow C_0(\mathbb{D}) \otimes C(S^1) \longrightarrow C(ST^2) \longrightarrow C(S^1) \otimes C(S^1) \longrightarrow 0$

APS Setup

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- Let M have a "product" structure near the boundary so that an infinite cylinder can be attached.
- Let D have a "special" decomposition structure so that it extends naturally to the infinite cylinder.

APS Boundary Condition

Study D with domain:

- $F \in H^1(M)$

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- $F \in H^1(M)$
- There is a $F^{\text{ext}} \in H^1_{\text{loc}}(cylinder)$ such that $DF^{\text{ext}} = 0$, $F^{\text{ext}}\big|_Y = F|_Y$ and $F^{\text{ext}} \in L^2(cylinder)$

Dirac Operator and Boundary Conditions

- $$D = \begin{pmatrix} \frac{1}{i}\frac{\partial}{\partial\theta} & 2\frac{\partial}{\partial\bar{z}} \\ -2\frac{\partial}{\partial z} & -\frac{1}{i}\frac{\partial}{\partial\theta} \end{pmatrix}$$

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- $$\mathrm{dom}(D) = \{F \in H^1(ST^2)\otimes\mathbb{C}^2 \,:\, \exists F^{ext} \in H^1_{loc}((\mathbb{C}\times S^1)\setminus ST^2)\otimes\mathbb{C}^2\}$$

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- $$(1)\ \ F^{ext}\big|_{\mathbb{T}^2} = F|_{\mathbb{T}^2}\,,\ (2)\ DF^{ext} = 0,\ (3)\ F^{ext} \in L^2((\mathbb{C}\times S^1)\setminus ST^2)\otimes\mathbb{C}^2$$

Fourier Decomposition

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- $$\|F\|^2 = \langle F, F \rangle = \sum_{m,n \in \mathbb{Z}} \int_0^1 \left(|f_{m,n}(r)|^2 + |g_{m,n}(r)|^2 \right) rdr$$

Kernel of D (without boundary conditions)

- $\mathrm{Ker}(D)$ consists of those functions $F \in L^2(ST^2 \setminus (\{0\} \times S^1)) \otimes \mathbb{C}^2$ such that

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\begin{aligned}
&m \neq 0, n \in \mathbb{Z} : \\
&f_{m,n+1}(r) = \frac{m}{|m|}\left(-A_{m,n} I_{n+1}(|m|r) + B_{m,n} K_{n+1}(|m|r)\right) \\
&g_{m,n}(r) = A_{m,n} I_n(|m|r) + B_{m,n} K_n(|m|r)
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- $\mathrm{Ker}(D)$ consists of those functions $F \in L^2(ST^2 \setminus (\{0\} \times S^1)) \otimes \mathbb{C}^2$ such that
- $$\begin{aligned} & m \neq 0, n \in \mathbb{Z} : \\ & f_{m,n+1}(r) = \frac{m}{|m|} \left(-A_{m,n} I_{n+1}(|m|r) + B_{m,n} K_{n+1}(|m|r)\right) \\ & g_{m,n}(r) = A_{m,n} I_n(|m|r) + B_{m,n} K_n(|m|r) \end{aligned}$$
- $m = 0,\ n \in \mathbb{Z}$: $f_{0,n}(r) = A_{0,n} r^{-n}$ and $g_{0,n}(r) = B_{0,n} r^n$

Boundary Condition Equivalence

- Let $F \in \mathrm{dom}(D)$, then

$$
\begin{aligned}
&|m|K_{n+1}(|m|)g_{m,n}(1) - mK_n(|m|)f_{m,n+1}(1) = 0 && m \neq 0, n \in \mathbb{Z} \\
&f_{0,n}(1) = 0 && m = 0, n \leq 0 \\
&g_{0,n}(1) = 0 && m = 0, n \geq 0
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- Subject to the boundary conditions, $\mathrm{Ker}(D) = \{0\}$ and $D^* = D$. Also there is an operator Q such that $QD = DQ = I$, that is $QDF = F$ for $F \in \mathrm{dom}(D)$ and $DQF = F$ for $F \in L^2(ST^2) \otimes \mathbb{C}^2$, i.e. D is invertible.

The Inverse I

- $$G = \sum_{m,n \in \mathbb{Z}} \begin{pmatrix} p_{m,n}(r) \\ q_{m,n}(r) \end{pmatrix} e^{in\varphi + im\theta}$$

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- $$G = \sum_{m,n\in\mathbb{Z}} \begin{pmatrix} p_{m,n}(r) \\ q_{m,n}(r) \end{pmatrix} e^{in\varphi+im\theta}$$
- $$QG := \sum_{m\in\mathbb{Z}\setminus\{0\},n\in\mathbb{Z}} \begin{pmatrix} f_{m,n}(r) \\ g_{m,n}(r) \end{pmatrix} e^{in\varphi+im\theta} + \sum_{n\in\mathbb{Z}} \begin{pmatrix} f_{0,n}(r) \\ g_{0,n}(r) \end{pmatrix} e^{in\varphi}$$

The Inverse II

for $m \neq 0$, let $\mathcal{K}_{i,j}(x,y) = mI_i(x)K_j(y)$

$$\begin{aligned} f_{m,n+1}(r) = & \int_r^1 |\mathcal{K}_{n+1,n}(|m|r,|m|\rho)|q_{m,n}(\rho)\rho d\rho \\ & + \int_r^1 \mathcal{K}_{n+1,n+1}(|m|r,|m|\rho)p_{m,n+1}(\rho)\rho d\rho \\ & - \int_0^r |\mathcal{K}_{n,n+1}(|m|\rho,|m|r)|q_{m,n}(\rho)\rho d\rho \\ & + \int_0^r \mathcal{K}_{n+1,n+1}(|m|\rho,|m|r)p_{m,n+1}(\rho)\rho d\rho \end{aligned}$$

The Inverse III

$$
\begin{aligned}
g_{m,n}(r) = &- \int_r^1 \mathcal{K}_{n,n}(|m|r, |m|\rho) q_{m,n}(\rho)\rho d\rho \\
&- \int_r^1 |\mathcal{K}_{n,n+1}(|m|r, |m|\rho)| p_{m,n+1}(\rho)\rho d\rho \\
&- \int_0^r \mathcal{K}_{n,n}(|m|\rho, |m|r) q_{m,n}(\rho)\rho d\rho \\
&+ \int_0^r |\mathcal{K}_{n+1,n}(|m|\rho, |m|r)| p_{m,n+1}(\rho)\rho d\rho
\end{aligned}
$$

The Inverse IV

for $m = 0$

$$f_{0,n}(r) = \begin{cases} -\int_0^r \frac{\rho^n}{r^{n+1}} q_{0,n}(\rho)\rho d\rho & n \geq 0 \\ \int_r^1 \frac{\rho^n}{r^{n+1}} q_{0,n}(\rho)\rho d\rho & n < 0 \end{cases}$$

$$g_{0,n}(r) = \begin{cases} -\int_r^1 \frac{r^n}{\rho^{n+1}} p_{0,n+1}(\rho)\rho d\rho & n \geq 0 \\ \int_0^r \frac{r^n}{\rho^{n+1}} p_{0,n+1}(\rho)\rho d\rho & n < 0 \end{cases}$$

Results

- **Theorem**
  The inverse Q of the Dirac operator defined by D, is bounded. Moreover Q is a compact operator, this means this is an elliptic boundary value problem.

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- **Theorem**
  The inverse Q of the Dirac operator defined by D, is bounded. Moreover Q is a compact operator, this means this is an elliptic boundary value problem.
- **Theorem**
  The inverse Q is a p-th Schatten-class operator for $p > 3$.

Bibliography

- Atiyah, M. F., Patodi, V. K. and Singer I. M., Spectral asymmetry and Riemannian geometry I. *Math. Proc. Camb. Phil. Soc.*, 77, 43 - 69, 1975.
- Klimek, S. and McBride, M., D-bar Operators on Quantum Domains. *Math. Phys. Anal. Geom.*, 13, 357 - 390, 2010.
- Klimek, S. and McBride, M., Global boundary conditions for a Dirac operator on the solid torus. *Jour. Math. Phys.*, 52, 1 - 14, 2011.

The End

Thank You