# RESEARCH STATEMENT

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## 1. Overview

I am an analyst specializing in functional analysis and operator theory. I am also interested in the theory of partial differential equations. My current work focuses in spectral theory of Schrödinger and Jacobi operators and noncommutative geometry. All of these areas stem back to mathematical physics and a lot of the work I have focused on has initially started as a mathematical physics problem. In fact in the noncommutative geometry area I am interested in Dirac and Dirac-type operators and their analysis classically and the analysis of them noncommutatively. This analysis also involves spectral analysis as well, which can tie into the spectral theory for Schrödinger and Jacobi operators. Moreover the analysis of partial differential equations relates to this theory, so I am also interested in the function space theory of such equations. The main areas of my interests: noncommutative geometry, partial differential equations, and spectral theory of Schrödinger and Jacobi operators, will have its own section ordered alphabetically.

## 2. Noncommutative Geometry

In the area of noncommutative geometry I am interested in quantum domains and quantum Dirac-type operators over these domains. In my thesis I described what certain noncommutative (quantum) spaces were and what the types of operators that act in these spaces were. When I tried to describe the space's structure, differential-difference operators arose naturally and I developed a theory for these operators. Since it is known how differential operators behave on known commutative spaces, one hopes there will be a similar theory in the quantum case. I was particularly interested in Dirac operators. Knowing how Dirac operators behaved classically, I wanted to construct their quantum analogs called quantum Dirac operators. It is natural to consider this construction to be some type of commutator. To be more specific, the form I used for a quantum Dirac operator $\delta$ was

$$\delta(a) = A[a, U_W]$$

where $U_W$ was either a bounded weighted unilateral shift or a bounded weighted bilateral shift and $A$ was an unbounded diagonal operator. Since the commutation with $U_W$ was bounded, the operator $A$ was introduced so that $\delta$ became an unbounded operator. I wanted $\delta$ to be unbounded since the Dirac operator in the classical case is unbounded.

One aspect of the theory of quantum Dirac operators is the index. The Atiyah-Patodi-Singer Index Theorem (APS) establishes an index formula for Dirac operators subject to the APS boundary condition on a closed manifold with boundary which depends only on the topological properties of the manifold. If $D$ is a Dirac or d-bar operator and $D^*$ is its

*Date*: October 9, 2014.

adjoint, then $DD^*$ and $D^*D$ are self adjoint and their non-zero eigenvalues have the same multiplicities. However their zero eigenspaces may have different multiplicities. Then it was shown in [1] that for $t \geq 0$

$$\text{Index}(D) := \dim \text{Ker}(D) - \dim \text{Ker}(D^*) = \text{tr}\left(e^{-tD^*D}\right) - \text{tr}\left(e^{-tDD^*}\right).$$

For this to be established, the authors first had to show that $D$ was a Fredholm operator, meaning that $D$ was closed and $D$ had closed range since $D$ was unbounded, and that $D$ also had finite dimensional kernel and cokernel. They were able to establish this by showing $D$ was invertible modulo compact operators which is an alternate way of showing an operator is Fredholm. This was when the APS boundary condition had to be initially setup. Let $M$ be a closed manifold with boundary that has a collar structure near the boundary so that an infinite (half-infinite) cylinder can be attached. In other words if $Y$ is the boundary of $M$, then $M$ has a product structure near $Y$ and one can consider $Y \times \mathbb{R}_{\geq 0}$. Moreover the authors of [1] required that $D$ was decomposed in the following way near the boundary:

$$D = \frac{\partial}{\partial t} + B$$

where $B$ was a first order self adjoint elliptic operator acting on $C^\infty(Y, E)$ and $E$ was a vector bundle over $Y$. This decomposition of $D$ extended naturally to $Y \times \mathbb{R}_{\geq 0}$. The APS boundary condition can be reformulated for $Y \times \mathbb{R}_{\geq 0}$. The condition was that for sections $f(y,t)$ of $E$ lifted to $Y \times \mathbb{R}_{\geq 0}$, one had

$$Pf(\cdot\ ,0) = 0 \tag{2.1}$$

where $P$ was the positive spectral projection of $B$. This is a non-local boundary condition that alleviates problems that classical boundary conditions, such as Dirichlet, have in global analysis. Since most of my current research has been spent finding inverses or inverses modulo compact operators to classical and quantum Dirac operators over different closed manifolds with boundary, I will quote the technical theorem that was established in [1]. The space of all $C^\infty$ functions satisfying (2.1) will be denoted by $C^\infty(Y \times \mathbb{R}_{\geq 0}, E; P)$ and $C^\infty_{\text{comp}}$ will denote functions vanishing for $t \geq C$ for some constant $C$. Also $H^k$ will denote the Sobolev space of sections with derivatives up to order $k$ in $L^2$.

**Theorem 2.1.** *There is a linear operator*

$$Q : C^\infty_{comp}(Y \times \mathbb{R}_{\geq 0}, E) \to C^\infty(Y \times \mathbb{R}_{\geq 0}, E; P)$$

*such that*

- *(i)* $DQg = g$ *for all* $g \in C^\infty_{comp}(Y \times \mathbb{R}_{\geq 0}, E)$
- *(ii)* $QDf = f$ *for all* $f \in C^\infty(Y \times \mathbb{R}_{\geq 0}, E; P)$
- *(iii) The kernel* $Q(y,t;z,v)$ *of* $Q$ *is* $C^\infty$ *for* $t \neq v; y, z \in Y$ *and* $t, v \in \mathbb{R}_{\geq 0}$
- *(iv)* $Q$ *extends to a continuous map* $H^{k-1} \to H^k_{loc}$ *for all integers* $k \geq 1$.

The main technique the authors used to prove this was expanding the solutions in terms of the eigenfunctions of $B$, in other words, they did a spectral/Fourier decomposition. As of now, there is no known generic index formula, or even a standard technique, like the above theorem, that generalizes the index theorem to noncommutative spaces. The goal of my work over the past years has been trying to generalize the above theorem into the noncommutative setup as well as generalize to other non-local boundary conditions and not just the APS condition. This goal was inspired by [1], but also by the presentation done in the book [2] and the treatments done in papers [3] and [11]. Throughout my papers and thesis, a general technique has worked for estimating the parametrices albeit it's restricted to specific domains and their quantum analogs. I believe this technique may work in general. The difficulty lies with figuring out how to solve the differential equation arising from the generic setup since using the Fourier decomposition technique will produce a higher order ordinary differential equation that is not easily solved. I believe that this can be gotten around by using standard techniques in the theory of partial differential equations by estimating solutions and there should be a corresponding theory for the difference equations that are produced in the quantum case. This is one of my short term goals.

The technique that was mentioned above was described in my papers with Slawomir Klimek. See [12] and [13] for details. In [12] we studied parametrices of the d-bar operator on the quantum disk and quantum annulus. Also the boundary conditions are not quite APS boundary conditions since the d-bar operator was not able to be decomposed into the special form that was required by APS. However the boundary condition studied in this paper was still a global condition. In [13] we were actually able to apply the honest APS boundary conditions because we studied $\overline{z}\overline{\partial}$ on the punctured disk which was able to be decomposed into the special form. Even though we could apply APS directly in [13], the operator we chose was not a natural one. Therefore a more generic global boundary condition that was inspired APS was necessary since the operators that naturally arose here do not have this special decomposition.

When a good formulation of a quantum disk and a quantum d-bar operator arose one of the natural questions was about the classical limit of such objects. Even more importantly, does the quantum d-bar operator converge, in some sense, to the classical d-bar operator? In [13], this question is answered in two domains, the disk and annulus. The question was answered through the parametrices to the d-bar operators via continuous fields of Hilbert spaces. This paper left the open question, “could this analysis be applied to other domains?” I hope to answer this question.

The papers [15] and [16] almost go hand in hand, though they can be thought of separately. In [15], we studied a Dirac operator on the classical solid torus subject to APS-like global boundary conditions. This boundary condition was partially inspired by the non-local condition described in [24]. Subtle analysis on the modified Bessel functions was necessary to show the parametrix to the Dirac operator was compact. Recently in [16], we studied a more generic Dirac operator on the quantum solid torus subject to global boundary condition. The global boundary condition here was developed from the one in [15] to suit the noncommutative structure in the quantum solid torus. Since the modified Bessel functions played an important role for the solutions in the classical torus paper, there were equally important quantum analogs to them that arose in the quantum torus paper. In the noncommutative setup, the analysis of these quantum analogs to the modified Bessel functions led to the analysis of continued fractions that had sparked interest to a separate paper not involving any type of Dirac operators, but just an interesting generic analysis paper.

In [18], Slawomir Klimek, Sumedha Rathnayake, Kaoru Sakai (students of Slawomir's), and myself studied different types of estimates of divergent continued fractions. The problem considered a region in the complex plane which contained values of the convergents of a continued fraction of a special type. More specifically in our "value region problem" we considered only even convergents for continued fractions of the Stieltjes type with bounded ratio of consecutive elements regardless of the convergence of the fractions. Even more generally, we also studied the same question for tail sequences and for (what we called) reverse sequences associated with a continued fraction. The main results we obtained in this paper showed that two types of circles with sufficiently large radii formed such value regions. Our interest in this problem was motivated by an unrelated study in [16], where considerations in noncommutative geometry required estimates on certain sequences that could be interpreted as tail and reverse sequences of a continued fraction. Here we looked at a far more general situation than needed for [16], namely we considered continued fractions whose elements were complex numbers with positive real part and a full value region problem.

Continuing on with classical and quantum domains, Slawomir and I, in [17], studied Dirac operators on the classical and quantum 2-sphere via gluing disks classically and noncommutatively. The concept of the quantum 2-sphere, sometimes called a mirror sphere, studied in [4] was the main reference. Specifically in the quantum case a generic Dirac type operator that was developed in [16] was studied. The boundary condition in this paper, however, was not really an APS type nor a global boundary condition but derived from the necessity on gluing the disks properly together. Nonetheless the boundary condition was still a valid and interesting local boundary condition. As with the other papers, the parametrices to the classical Dirac and quantum Dirac type operators were studied and they were shown to be compact operators.

Another project that Slawomir, Sumedha and myself also worked on considered spectral triples continuing the program of quantum domains. This project considered when the boundary was the Cantor set and the manifold was a discrete set, namely it was a tree and we construct a spectral triple. The idea of this was inspired by John Pearson's PhD. thesis at Georgia Tech [25]. In our paper [19], we constructed a spectral triple for the $C^*$-algebra of continuous functions on the space of $p$-adic integers by using a rooted tree obtained from coarse-grained approximation of the space, and the forward derivative on the tree. Additionally, we verified that our spectral triple satisfies the properties of a compact spectral metric space, and we showed that the metric on the space of $p$-adic integers induced by the spectral triple was equivalent to the usual $p$-adic metric.

In fact, we are currently computing the spectral triples of all quantum domains Slawomir and I studied, namely the quantum annulus, quantum punctured disk, quantum solid torus, and the quantum 2-sphere.

In the ongoing process of generalizing quantum domains, Slawomir, Sumedha, Kaoru, and I constructed a $C^*-$algebra that is a good candidate to be defined as the quantum pair of pants, i.e. the quantum disk with two holes. In [20] we computed the spectrum of the operator of multiplication by the complex coordinate in a Hilbert space of holomorphic functions on a disk with two circular holes. We also determined the structure of the $C^*-$algebra generated by that operator. This algebra can be considered as the quantum pair of pants. The case of a disk with no holes is the classical one. In the Hardy space of the disk the multiplication operator $z$ is the unilateral shift whose spectrum is the disk. The $C^*-$algebra generated by the unilateral shift, called the Toeplitz algebra, is an extension of the algebra of compact operators by $C(S^1)$, $S^1$ being the boundary of the disk. For the Bergman space the $z$ operator is a weighted unilateral shift and its spectrum and the $C^*-$algebra it generates are the same as in the Hardy space. Partially for those reasons the Toeplitz algebra is often considered as the quantum disk. We then followed the idea to construct what we called the quantum pair of pants.

In [23], I was able to generalize the quantum pair of pants to the quantum $m$-legged pair of pants, i.e. the quantum disk with $m$ holes with arbitrary centers. I followed the same idea in [20] and constructed a $C^*-$algebra that is a reasonable candidate to be termed the quantum $m$-legged pair of pants.

Following suit of the analysis of quantum Dirac-type operators I am currently trying to develop the same kind of analysis for these operators on the quantum pair of pants in spirit of the papers [12]-[17], excluding [14]. As in the other papers, I am trying to construct a parametrix in both the classical case and quantum case subject to APS like boundary conditions and show that they are compact yielding an elliptic boundary value problem. Developing a similar kind of analysis for these operators on the quantum $m$-legged pairs of pants will be a natural project in the future.

Future research that Slawomir and myself plan to work on is trying to develop a way to glue two noncommutative manifolds in a noncommutative way. We are interested in this since we think this could be a way in trying to make noncommutative analog of the APS index theorem.

## 3. Partial Differential Equations

In the area of partial differential equations, I am interested in regularity conditions, namely the area of elliptic and parabolic PDEs. In [6], Huang showed that in the elliptic system $D_\alpha(a_{ij}^{\alpha\beta} D_\beta u^j) = -\text{ div } f^i$, the solutions $u^j(x)$, for $x \in \mathbb{R}^n$ satisfied some regularity estimate. In particular two of the things shown were that first, if $a_{ij}^{\alpha\beta} \in C(B_R)$ then $Du \in L_\varphi^{2,\lambda}$ , and secondly if $a_{ij}^{\alpha\beta} \in VMO(B_R)$ then $Du \in L_\varphi^{2,\lambda}$. Here $B_R$ is the $n-$dimensional ball of radius $R$ centered at $x_0$ and $L_\varphi^{2,\lambda}$ was defined to be the Morrey space. More specifically the Morrey space is

$$L_\varphi^{p,\lambda}(B_R) = \left\{ f \in L^p(B_R) \ : \ \left( \sup_{y_0 \in B_R, 0 \le \rho \le d} \frac{1}{\varphi(\rho)} \rho^{-\lambda} \int_{B_R \cap B_\rho(y_0)} |f|^p \, dx \right)^{\frac{1}{p}} < \infty \right\}$$

with $1 \le p < \infty$, $0 \le \lambda \le n+2$, and $\varphi$ is a continuous function on $[0,d]$, $\varphi > 0$ on $(0,d]$, and $d$ is the diameter of $B_R$. Moreover the $VMO$ space is just the vanishing mean oscillation space. In [22], I expanded these two facts in a short paper for a master's thesis. I was able to show these results to the parabolic case where now the parabolic system I studied was $u_t^i - D_\alpha(a_{ij}^{\alpha\beta} D_\beta u^j) = -\text{ div } f^i$ over the generalized Morrey Space $L_\varphi^{2,\lambda}$.

This is the area that I have done few things in. However I would like to expand my knowledge in this area of PDEs and develop theories based around them. Specifically one future project would be to go back and expand the rest of the theory developed in [6] to the parabolic case since to my knowledge this has yet to be done. Other future projects include studying other elliptic and parabolic equations and expanding other regularity estimates that are not already known.

## 4. Spectral Theory of Schrödinger and Jacobi operators

Finally in the spectral theory area I am working with Christian Remling and Injo Hur. We are interested in one-dimensional Schrödinger operators,

$$L = -\frac{d^2}{dx^2} + V(x), \tag{4.1}$$

with locally integrable potentials $V$ that are in the limit point case at $\pm\infty$ and in Jacobi matrices,

$$(Ju)_n = a_n u_{n+1} + a_{n-1}u_{n-1} + b_n u_n. \tag{4.2}$$

Here we assume that $a, b \in \ell^\infty(\mathbb{Z})$, $a_n > 0$, $b_n \in \mathbb{R}$.

These operators have associated half line $m$ functions $m_\pm$. These are *Herglotz functions,* that is, they map the upper half plane $\mathbb{C}^+$ holomorphically to itself.

We call an operator *reflectionless* on a Borel set $S \subset \mathbb{R}$ of positive Lebesgue measure if $m_\pm$ satisfy the following identity

$$m_+(x) = -\overline{m_-(x)} \qquad \text{for (Lebesgue) a.e. } x \in S. \tag{4.3}$$

Reflectionless operators are important because they can be thought of as the fundamental building blocks of arbitrary operators with some absolutely continuous spectrum. See [8, 26, 27]. Reflectionless operators have remarkable properties, and if an operator is reflectionless on an *interval* (rather than a more complicated set), one can say even more. So these operators are of special interest.

Marchenko [21] developed a certain parametrization of the class $\mathcal{M}_R$ of Schrödinger operators $H$ that were reflectionless on $(0, \infty)$ and had spectrum contained in $[-R^2, \infty)$ (this is paraphrasing as Marchenko did not emphasize this aspect, and his goals were different from ours). It is in fact easy in principle to give such a parametrization in terms of certain spectral data, which has been used by many authors see [27] or [29] for example. Marchenko's parametrization was different, and it made certain properties of reflectionless Schrödinger operators very transparent.

In [7], we had two general goals: First, we presented a direct and easy approach to Marchenko's parametrization that started from scratch and did not use any inverse scattering theory which Marchenko's treatment had relied on as its main tool (which then needed to be combined with a limiting process, as most reflectionless operators do not fall under the scope of classical scattering theory) and was rather intricate. Marchenko had inequalities that were imposed on the representing measures $\sigma$ that were never fully addressed. In our paper we explained the role of these inequalities. We also extended these ideas to the discrete setting; in fact, we started with this case as some technical issues from the continuous setting were absent here. The second goal was to explore some consequences and applications of Marchenko's parametrization, in the form developed in our paper [7].

Currently we are studying canonical systems. A canonical system is a family of differential equations of the form

$$Au'(x) = zH(x)u(x) \text{ for } z \in \mathbb{C} \text{ and } A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $H(x)$ is a positive semidefinite matrix. These can be related to Schrödinger equations and transfer matrices. Transfer matrices can be used to solve these canonical systems. A lot of the theory for transfer matrices is very difficult to understand. At the moment we analyzing the theory and trying to see if we can recreate it from scratch to see if there are easier and more straightforward methods. Christian has a plan to write a book over the spectral theory of Schrödinger and Jacobi operators and this is something he would like to include if it turns out the theory can be simplified.

Another project that we recently started was the inverse spectral theory for one-dimensional periodic Schrödinger operators and periodic Jacobi operators. In [5], the authors study the operator

$$L = -\frac{d^2}{dx^2} + V(x)$$

with potential $V(x) = -4\alpha \cos 2x - 2\alpha^2 \cos 4x$ where $\alpha$ and $s$ are real and natural numbers respectively. When written as an eigenvalue problem, this is known as the Wittaker-Hill equation. They used a specific trigonometric substitution and transform the equation into a seemingly more difficult problem but were able to give criteria for the regularity of the corresponding potentials and described the spectral properties more in detail, namely when the gaps are closed. This is only for a specific problem, but since we are interested in the reverse direction, we are using this as a starting example.

## 5. Conclusion

In the future I plan to continue to develop the theory of differential and difference operators that could have various applications to differential/difference equation theory, spectral theory, and just general solution methods and expansion theorems for those operators acting on a particular space. I feel that developing these types of expansion theorems will give more general results about solutions to differential equations, partial and ordinary, since they will be treated as a differential operator acting on some space of functions. I also believe that in studying spectral theory for differential/difference operators can have rich results since not much is known in full generality. These are just some of my long term and short term goals in mathematics. I feel research is a long term learning process and can develop into different areas of mathematics that one may have thought had no relation to current research. This is why I like to always learn new things even if it is from different areas.

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