The Marchenko representation of reflectionless Jacobi and Schrodinger operators

Matt McBride

University of Oklahoma

March 1, 2014

Introduction and Relavent Papers

- This is joint work with Injo Hur and Christian Remling.

Introduction and Relavent Papers

- This is joint work with Injo Hur and Christian Remling.
- Hur, I., McBride, M. and Remling, C., The Marchenko Representation of Reflectionless Jacobi and Schrödinger Operators. arXiv:1401.7704. submitted to Trans. AMS.

Introduction and Relavent Papers

- This is joint work with Injo Hur and Christian Remling.
- Hur, I., McBride, M. and Remling, C., The Marchenko Representation of Reflectionless Jacobi and Schrödinger Operators. arXiv:1401.7704. submitted to Trans. AMS.
- Marchenko, V., The Cauchy Problem for the KdV Equation with Non-decreasing Initial Data. *What is Integrability?*, 273 - 318, Springer, Berlin, 1991.

Introduction and Relavent Papers

- This is joint work with Injo Hur and Christian Remling.
- Hur, I., McBride, M. and Remling, C., The Marchenko Representation of Reflectionless Jacobi and Schrödinger Operators. arXiv:1401.7704. submitted to Trans. AMS.
- Marchenko, V., The Cauchy Problem for the KdV Equation with Non-decreasing Initial Data. *What is Integrability?*, 273 - 318, Springer, Berlin, 1991.
- Remling, C., The Absolutely Continuous Spectrum of Jacobi Matrices. *Annals of Math.*, 174, 125 - 171, 2011.

Korteweg-de Vries Equation

- $u_t = 6uu_x - u_{xxx}$

Korteweg-de Vries Equation

- $u_t = 6uu_x - u_{xxx}$
- Can transform this PDE into the following Lax pair

Korteweg-de Vries Equation

- $u_t = 6uu_x - u_{xxx}$
- Can transform this PDE into the following Lax pair
- $$\frac{dL}{dt} = [P, L]$$

Korteweg-de Vries Equation

- $u_t = 6uu_x - u_{xxx}$
- Can transform this PDE into the following Lax pair
- $$\frac{dL}{dt} = [P, L]$$
- where $L = -D^2 + u$, $P = -4D^3 + 3(uD + Du)$, and $D = \partial/\partial x$

Korteweg-de Vries Equation

- $u_t = 6uu_x - u_{xxx}$
- Can transform this PDE into the following Lax pair
- $$\frac{dL}{dt} = [P, L]$$
- where $L = -D^2 + u$, $P = -4D^3 + 3(uD + Du)$, and $D = \partial/\partial x$
- Studying the KdV equation will amount to studying the operator L.

One-dimensional Schrödinger Operators

- Want to study:

$$L = -\frac{d^2}{dx^2} + V(x)$$

One-dimensional Schrödinger Operators

- Want to study:

$$L = -\frac{d^2}{dx^2} + V(x)$$

- on $L^2(\mathbb{R})$, where $V \in L^1_{\text{loc}}(\mathbb{R})$ and have limit point case at $\pm\infty$.

One-dimensional Schrödinger Operators

- Want to study:

$$L = -\frac{d^2}{dx^2} + V(x)$$

- on $L^2(\mathbb{R})$, where $V \in L^1_{\text{loc}}(\mathbb{R})$ and have limit point case at $\pm\infty$.
- In general, $Ly = \lambda y$ with a condition at 0, Dirichlet for example, will yield limit points or limit circles, known as Weyl circles.

Limit points and Limit Circles

Study D with domain:

- Given an interval (a, b),

Limit points and Limit Circles

Study D with domain:

- Given an interval (a, b),
- L is in the limit circle case at a if there is a $u \in \mathrm{dom}(L)$ with $W(u^*, u)(a) = 0$ such that $W(u, f) \neq 0$ for at least one $f \in \mathrm{dom}(L)$.

Limit points and Limit Circles

Study D with domain:

- Given an interval (a, b),
- L is in the limit circle case at a if there is a $u \in \mathrm{dom}(L)$ with $W(u^*, u)(a) = 0$ such that $W(u, f) \neq 0$ for at least one $f \in \mathrm{dom}(L)$.
- Otherwise we call L in the limit point case.

Weyl-Titchmarsh Functions I

- For $\lambda \in \mathbb{C}^+$, assuming the limit point case for L, Dirichlet condition at 0, and the other conditions on L from before, there are unique solutions $u_\pm$, up to a constant factor, of

$$Ly = \lambda y$$

such that $u_\pm \in L^2$ near $\pm\infty$ respectively.

Weyl-Titchmarsh Functions I

- For $\lambda \in \mathbb{C}^+$, assuming the limit point case for L, Dirichlet condition at 0, and the other conditions on L from before, there are unique solutions $u_\pm$, up to a constant factor, of

$$Ly = \lambda y$$

  such that $u_\pm \in L^2$ near $\pm\infty$ respectively.
- This is also the domain of L.

Weyl-Titchmarsh Functions I

- For $\lambda \in \mathbb{C}^+$, assuming the limit point case for L, Dirichlet condition at 0, and the other conditions on L from before, there are unique solutions $u_\pm$, up to a constant factor, of

$$Ly = \lambda y$$

such that $u_\pm \in L^2$ near $\pm\infty$ respectively.
- This is also the domain of L.
- Define the Weyl-Titchmarsh $m-$functions as

$$m_\pm(\lambda) = \frac{u'_\pm(0,\lambda)}{u_\pm(0,\lambda)}$$

Weyl-Titchmarsh Functions II

- Can think of these functions as splitting L into two parts, L_- corresponding to $(-\infty, 0)$, and L_+ corresponding to $(0, \infty)$ and thinking them as a matrix

Weyl-Titchmarsh Functions II

- Can think of these functions as splitting L into two parts, L_- corresponding to $(-\infty, 0)$, and L_+ corresponding to $(0, \infty)$ and thinking them as a matrix
- Then studying the resolvent set on the diagnoal $\langle \delta_0, (L_\pm - \lambda)^{-1} \delta_0 \rangle$

Weyl-Titchmarsh Functions II

- Can think of these functions as splitting L into two parts, L_- corresponding to $(-\infty, 0)$, and L_+ corresponding to $(0, \infty)$ and thinking them as a matrix
- Then studying the resolvent set on the diagnoal $\langle \delta_0, (L_\pm - \lambda)^{-1} \delta_0 \rangle$
- Which can also be thought of as a Green's function.

Weyl-Titchmarsh Functions II

- Can think of these functions as splitting L into two parts, L_- corresponding to $(-\infty, 0)$, and L_+ corresponding to $(0, \infty)$ and thinking them as a matrix
- Then studying the resolvent set on the diagnoal $\langle \delta_0, (L_\pm - \lambda)^{-1} \delta_0 \rangle$
- Which can also be thought of as a Green's function.
- $m_\pm(\lambda)$ has the following asymptotics as $|\lambda| \to \infty$ in $\varepsilon < \arg\lambda < \pi - \varepsilon$

$$m_\pm(\lambda) = \sqrt{-\lambda} + o(1)$$

Reflectionless Operators

- L is said to be reflectionless on a Borel set $S \subset \mathbb{R}$ with $m(S) > 0$ if its $m-$functions satisfy the following:

Reflectionless Operators

- L is said to be reflectionless on a Borel set $S \subset \mathbb{R}$ with $m(S) > 0$ if its $m-$functions satisfy the following:
- $$m_+(x) = -\overline{m_-(x)}$$

  for a.e. $x \in S$.

Reflectionless Operators

- L is said to be reflectionless on a Borel set $S \subset \mathbb{R}$ with $m(S) > 0$ if its $m-$functions satisfy the following:
- $$m_+(x) = -\overline{m_-(x)}$$

  for a.e. $x \in S$.
- Define the following set

$$\mathcal{M}_R = \left\{L : L \text{ is reflectionless on } (0,\infty) \text{ and } \sigma(L) \subset [-R^2,\infty)\right\}$$

Herglotz Functions

- Recall a function $f(\lambda) : \mathbb{C}^+ \to \mathbb{C}^+$ which is holomorphic is called a Herglotz function and has the following representation:

Herglotz Functions

- Recall a function $f(\lambda) : \mathbb{C}^+ \to \mathbb{C}^+$ which is holomorphic is called a Herglotz function and has the following representation:
- $$f(\lambda) = a + b\lambda + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) \, d\sigma(t)$$

  a and b are real, and $b > 0$, σ nonzero measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} (1+t^2)^{-1} \, d\sigma(t) < \infty$.

Some Preperation

- For a reflectionless L, we get the following function:

Some Preperation

- For a reflectionless L, we get the following function:
-
$$M(\lambda) = \begin{cases} m_+(\lambda) & \lambda \in \mathbb{C}^+ \\ -\overline{m_-(\overline{\lambda})} & \lambda \in \mathbb{C}^- \end{cases}$$

Some Preperation

- For a reflectionless L, we get the following function:
- $$M(\lambda) = \begin{cases} m_+(\lambda) & \lambda \in \mathbb{C}^+ \\ -\overline{m_-(\overline{\lambda})} & \lambda \in \mathbb{C}^- \end{cases}$$
- This $M : \Omega \to \mathbb{C}^+$ is holomophic where $\Omega = \mathbb{C}^+ \cup (0, \infty) \cup \mathbb{C}^-$

Some Preperation

- For a reflectionless L, we get the following function:
- $$M(\lambda) = \begin{cases} m_+(\lambda) & \lambda \in \mathbb{C}^+ \\ \overline{-m_-(\bar{\lambda})} & \lambda \in \mathbb{C}^- \end{cases}$$
- This $M : \Omega \to \mathbb{C}^+$ is holomophic where $\Omega = \mathbb{C}^+ \cup (0, \infty) \cup \mathbb{C}^-$
- Using the conformal map $\varphi : \mathbb{C}^+ \to \Omega$ given by $\varphi(\lambda) = -\lambda^2$, we get

Some Preperation

- For a reflectionless L, we get the following function:
- $$M(\lambda) = \begin{cases} m_+(\lambda) & \lambda \in \mathbb{C}^+ \\ -\overline{m_-(\overline{\lambda})} & \lambda \in \mathbb{C}^- \end{cases}$$
- This $M : \Omega \to \mathbb{C}^+$ is holomophic where $\Omega = \mathbb{C}^+ \cup (0, \infty) \cup \mathbb{C}^-$
- Using the conformal map $\varphi : \mathbb{C}^+ \to \Omega$ given by $\varphi(\lambda) = -\lambda^2$, we get
- the following Herglotz function $F(\lambda) = M(\varphi(\lambda))$.

Main Theorem (Continuous Case)

Theorem

$L \in \mathcal{M}_R$ iff the associated $F-$function is of the form

$$F(\lambda) = \lambda + \int_{\mathbb{R}} \frac{d\sigma(t)}{t - \lambda}$$

for some finite Borel measure σ on $(-R, R)$ such that

$$1 + \int_{\mathbb{R}} \frac{d\sigma(t)}{t^2 - R^2} \geq 0$$

Moreover if $L \in \mathcal{M}_R$, then V is real analytic. More specifically $V(x)$ has a holomophic continuation $V(z)$ to the strip $|Im\ z| < 1/R$.

Jacobi Operators

- Now interested in Jacobi operators/matrices

Jacobi Operators

- Now interested in Jacobi operators/matrices
- Recall a Jacobi operator on a numerical function $u \in \ell^p(\mathbb{Z})$ is

$$(Ju)_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n$$

where a and b are in $\ell^\infty(\mathbb{Z})$, $a_n > 0$, and $b_n \in \mathbb{R}$.

Jacobi Operators

- Now interested in Jacobi operators/matrices
- Recall a Jacobi operator on a numerical function $u \in \ell^p(\mathbb{Z})$ is

$$(Ju)_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n$$

where a and b are in $\ell^\infty(\mathbb{Z})$, $a_n > 0$, and $b_n \in \mathbb{R}$.

- We can again study the equation $Ju = \lambda u$ with $u \in \ell^2(\mathbb{Z})$.

Reflectionless Jacobi Operators

- Now we want J to be reflectionless on $(-2, 2)$. The definiton for reflectionless is the same, but the $m-$functions here are different

Reflectionless Jacobi Operators

- Now we want J to be reflectionless on $(-2, 2)$. The definiton for reflectionless is the same, but the $m-$functions here are different
- For various reasons, it's necessary that $\|J\| \leq R$ for some $R \geq 2$. Want to study this J now on $\ell^2(\mathbb{Z})$.

Reflectionless Jacobi Operators

- Now we want J to be reflectionless on $(-2, 2)$. The definiton for reflectionless is the same, but the $m-$functions here are different
- For various reasons, it's necessary that $\|J\| \leq R$ for some $R \geq 2$. Want to study this J now on $\ell^2(\mathbb{Z})$.
- We have a similar $\mathcal{M}$ space like in the continuous case.

Reflectionless Jacobi Operators

- Now we want J to be reflectionless on $(-2, 2)$. The definiton for reflectionless is the same, but the $m-$functions here are different
- For various reasons, it's necessary that $\|J\| \leq R$ for some $R \geq 2$. Want to study this J now on $\ell^2(\mathbb{Z})$.
- We have a similar $\mathcal{M}$ space like in the continuous case.
- Let

$$\mathcal{M}_R = \{J : J \text{ is reflectionless on } (-2, 2) \text{ for } R \geq 2\}$$

$m-$Functions Again

- For $\lambda \in \mathbb{C}^+$, let $u_\pm(n,\lambda)$ be the two solutions to

$$a_n u(n+1,\lambda) + a_{n-1} u(n-1,\lambda) + b_n u(n,\lambda) = \lambda u(n,\lambda)$$

that are square summable near $\pm\infty$ respectively

$m-$Functions Again

- For $\lambda \in \mathbb{C}^+$, let $u_\pm(n, \lambda)$ be the two solutions to

$$a_n u(n+1, \lambda) + a_{n-1} u(n-1, \lambda) + b_n u(n, \lambda) = \lambda u(n, \lambda)$$

  that are square summable near $\pm\infty$ respectively
- The fact that $\|J\| < R$ guarantees these are unique solutions up to a factor

$m-$Functions Again

- For $\lambda \in \mathbb{C}^+$, let $u_\pm(n, \lambda)$ be the two solutions to

$$a_n u(n+1, \lambda) + a_{n-1} u(n-1, \lambda) + b_n u(n, \lambda) = \lambda u(n, \lambda)$$

  that are square summable near $\pm\infty$ respectively
- The fact that $\|J\| < R$ guarantees these are unique solutions up to a factor
- The $m-$functions are then defined to be

$$m_\pm(\lambda) = \mp \frac{u_\pm(1, \lambda)}{a_0 u_\pm(0, \lambda)}$$

More Herglotz

- Since J is reflectionless, we again get the function:

More Herglotz

- Since J is reflectionless, we again get the function:
- $$M(\lambda) = \begin{cases} m_+(\lambda) & \lambda \in \mathbb{C}^+ \\ -\overline{m_-(\overline{\lambda})} & \lambda \in \mathbb{C}^- \end{cases}$$

More Herglotz

- Since J is reflectionless, we again get the function:
- $$M(\lambda) = \begin{cases} m_+(\lambda) & \lambda \in \mathbb{C}^+ \\ -\overline{m_-(\overline{\lambda})} & \lambda \in \mathbb{C}^- \end{cases}$$
- This $M : \Omega \to \mathbb{C}^+$ is again holomophic where now $\Omega = \mathbb{C}^+ \cup (-2, 2) \cup \mathbb{C}^-$

More Herglotz

- Since J is reflectionless, we again get the function:
- $$M(\lambda) = \begin{cases} m_+(\lambda) & \lambda \in \mathbb{C}^+ \\ -\overline{m_-(\bar{\lambda})} & \lambda \in \mathbb{C}^- \end{cases}$$
- This $M : \Omega \to \mathbb{C}^+$ is again holomophic where now $\Omega = \mathbb{C}^+ \cup (-2, 2) \cup \mathbb{C}^-$
- Using the conformal map $\varphi : \mathbb{C}^+ \to \Omega$ given by $\varphi(\lambda) = -\lambda - \lambda^{-1}$ we get

More Herglotz

- Since J is reflectionless, we again get the function:
- $$M(\lambda) = \begin{cases} m_+(\lambda) & \lambda \in \mathbb{C}^+ \\ -\overline{m_-(\overline{\lambda})} & \lambda \in \mathbb{C}^- \end{cases}$$
- This $M : \Omega \to \mathbb{C}^+$ is again holomophic where now $\Omega = \mathbb{C}^+ \cup (-2, 2) \cup \mathbb{C}^-$
- Using the conformal map $\varphi : \mathbb{C}^+ \to \Omega$ given by $\varphi(\lambda) = -\lambda - \lambda^{-1}$ we get
- the following Herglotz function $F(\lambda) = M(\varphi(\lambda))$.

More Herglotz

- Since J is reflectionless, we again get the function:
- $$M(\lambda) = \begin{cases} m_+(\lambda) & \lambda \in \mathbb{C}^+ \\ -\overline{m_-(\bar{\lambda})} & \lambda \in \mathbb{C}^- \end{cases}$$
- This $M : \Omega \to \mathbb{C}^+$ is again holomophic where now $\Omega = \mathbb{C}^+ \cup (-2, 2) \cup \mathbb{C}^-$
- Using the conformal map $\varphi : \mathbb{C}^+ \to \Omega$ given by $\varphi(\lambda) = -\lambda - \lambda^{-1}$ we get
- the following Herglotz function $F(\lambda) = M(\varphi(\lambda))$.
- It should be noted $\varphi : S^+ \to (-2, 2)$, $\varphi : \mathbb{D}^+ \to \mathbb{C}^+$, and $\varphi : (\mathbb{D}^+)^c \to \mathbb{C}^-$.

Some Setup

- let r be the solution to $r + 1/r = R$ with $0 < r \leq 1$ (Well defined as $R \geq 2$).

Some Setup

- let r be the solution to $r + 1/r = R$ with $0 < r \leq 1$ (Well defined as $R \geq 2$).
- Also denote
$$\sigma_n = \int_{\mathbb{R}} t^n \, d\sigma(t)$$
the moments of the measure σ for $n \in \mathbb{Z}$.

Main Theorem (Discrete Case)

Theorem

$J \in \mathcal{M}_R$ iff the associated $F-$function is of the form

$$F(\lambda) = -\sigma_{-1} + (1 - \sigma_{-2})\lambda + \int_{\mathbb{R}} \frac{d\sigma(t)}{t - \lambda}$$

for some finite Borel measure σ on $(-1/r, -r) \cup (r, 1/r)$ such that

$$1 - \sigma_{-2} + \int_{\mathbb{R}} \frac{d\sigma(t)}{t^2 + ct + 1} > 0$$

for all $|c| > R$.

Meaning

- These theorems say if we have L or J, then they determine their respective $m-$functions and thus their respective $F-$functions and measures σ

Meaning

- These theorems say if we have L or J, then they determine their respective $m-$functions and thus their respective $F-$functions and measures σ
- On the otherhand, if we have a measure σ satisfying the conditions of either theorem, we can define an $F-$function and thus completely determine a unique L or J.

Meaning

- These theorems say if we have L or J, then they determine their respective $m-$functions and thus their respective $F-$functions and measures σ
- On the otherhand, if we have a measure σ satisfying the conditions of either theorem, we can define an $F-$function and thus completely determine a unique L or J.
- In otherwords, we can uniquely determine an operator of these types knowing only the associated measure satisfying some conditions.

Bibliography

- Hur, I., McBride, M. and Remling, C., The Marchenko Representation of Reflectionless Jacobi and Schrödinger Operators. arXiv:1401.7704. Submitted to Trans. AMS.
- Marchenko, V., The Cauchy Problem for the KdV Equation with Non-decreasing Initial Data. *What is Integrability?*, 273 - 318, Springer, Berlin, 1991.
- Remling, C., The Absolutely Continuous Spectrum of Jacobi Matrices. *Annals of Math.*, 174, 125 - 171, 2011.

The End

Thank You