D-bar Operators in Quantum Domains

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Two equivalent definitions

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- *Definition:* A closed operator D is said to be an unbounded Fredholm operator if there exists a bounded operator Q such that $DQ - I$ and $QD - I$ are compact.

Classical Case

- We define the disk as follows:

$$\begin{aligned} \mathbb{D} &= \{z \in \mathbb{C} \;:\; |z| \leq \rho\} \\ \partial\mathbb{D} &= \{z \in \mathbb{C} \;:\; |z| = \rho\} \simeq S^1 \end{aligned} \tag{3.1}$$

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- We define the annulus as follows:

$$\mathbb{A}_{\rho_-,\rho_+} = \{z \in \mathbb{C} \ : \ 0 < \rho_- \leq |z| \leq \rho_+\} \\ \partial\mathbb{A}_{\rho_-,\rho_+} = \{z \in \mathbb{C} \ : \ |z| = \rho_\pm\} \simeq S^1 \cup S^1 \tag{3.2}$$

Classical Case: Disk and Annulus short exact sequences

- Let D be the following operator:

$$D = \frac{\partial}{\partial \overline{z}} \qquad (3.3)$$

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$$\begin{gathered} 0 \longrightarrow C_0^\infty(\mathbb{D}) \longrightarrow C^\infty(\mathbb{D}) \xrightarrow{r} C^\infty(\partial\mathbb{D}) \longrightarrow 0 \\ 0 \longrightarrow C_0^\infty(\mathbb{A}_{\rho_-,\rho_+}) \longrightarrow C^\infty(\mathbb{A}_{\rho_-,\rho_+}) \xrightarrow{r=r_-\oplus r_+} \\ \xrightarrow{r=r_-\oplus r_+} C^\infty(S^1) \oplus C^\infty(S^1) \longrightarrow 0 \end{gathered} \tag{3.4}$$

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- Here r is the restriction to the boundary and $C_0^\infty(\cdot) = C^\infty(\cdot) \cap C_0(\cdot)$.

Classical Case: APS boundary conditions

- Let $\pi_A(I)$ be the spectral projection of a self-adjoint operator, A, onto an interval I. Let

$$\begin{aligned} P_N &= \pi_{\frac{1}{i}\frac{\partial}{\partial\varphi}}(-\infty, N] \quad N \in \mathbb{Z} \\ P_N^{\pm} &= \pi_{\pm\frac{1}{i}\frac{\partial}{\partial\varphi}}(-\infty, N] \quad N \in \mathbb{Z} \end{aligned} \tag{3.5}$$

Classical Case: Definition of D_N and $D_{M,N}$

- Let D_N be the operator D with domain

$$\mathrm{dom}(D_N) = \{f \in C^\infty(\mathbb{D}) \subset L^2(\mathbb{D}) \ : \ rf \in \mathrm{Ran} P_N\} \quad (3.6)$$

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- Let $D_{M,N}$ be the operator D with domain

$$\mathrm{dom}(D_{M,N}) = \{f \in C^\infty(\mathbb{A}_{\rho_-,\rho_+}) \subset L^2(\mathbb{A}_{\rho_-,\rho_+}) \ : \\ r_+ f \in \mathrm{Ran} P_M^+, \ r_- f \in \mathrm{Ran} P_N^-\} \tag{3.7}$$

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  The closure of D_N is an unbounded Fredholm operator in $L^2(\mathbb{D})$ and $ind(D_N) = N + 1$.

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- We also need the shift and diagonal operator defined respectively

$$\begin{aligned} Ue_k &= e_{k+1} \\ \Lambda e_k &= w_k e_k \end{aligned} \tag{4.2}$$

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- S defined above is injective.

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- Here r represents the symbol map, $r(I) = 1$ and $r(W) = e^{i\varphi}$.

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- For $a \in \mathrm{Pol}(W)$ define

$$\begin{aligned} D &: \mathrm{Pol}(W) \longrightarrow \mathcal{H} \\ Da &= S^{-1}[a, W] \end{aligned} \tag{4.4}$$

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- The above suggests that D looks like $\frac{\partial}{\partial \overline{z}}$, except for the non-commutativity.

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- Let $D_{M,N}$ be the operator D with domain

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The End

Thank You