Homework 3 Solutions

(1) $\lim_{x\to 2} f = 4$ $\quad \lim_{x\to 2} g = -2$ $\quad \lim_{x\to 2} h(x) = 0$

(a) $\lim_{x\to 2}[f(x)+5g(x)] = \lim_{x\to 2} f(x) + 5\lim_{x\to 2} g(x)$

$= 4+5(-2)$

$= 4-10$

$= \boxed{-6}$

(b) $\lim_{x\to 2}[g(x)]^3 = \left[\lim_{x\to 2} g(x)\right]^3$

$= (-2)^3$

$= \boxed{-8}$

(c) $\lim_{x\to 2}\sqrt{f(x)} = \sqrt{\lim_{x\to 2} f(x)}$

$= \sqrt{4}$

$= \boxed{2}$

(d) $\lim_{x\to 2}\dfrac{3f(x)}{g(x)} = \dfrac{3\lim_{x\to 2} f(x)}{\lim_{x\to 2} g(x)}$

$= \dfrac{3(4)}{-2}$

$= \boxed{-6}$

(e) $\lim_{x\to 2}\dfrac{g(x)}{h(x)} = \dfrac{\lim_{x\to 2} g(x)}{\lim_{x\to 2} h(x)}$

$= \dfrac{-2}{0}$

DNE because denominator's limit is 0.

(f) $\lim_{x\to 2} \frac{g(x)h(x)}{f(x)} = \frac{\lim_{x\to 2} g(x) \lim_{x\to 2} h(x)}{\lim_{x\to 2} f(x)}$

$= \frac{(-2)(0)}{4}$

$= \frac{0}{4}$

$= \boxed{0}$

2. $\lim_{x\to -1} (x^4 - 3x)(x^2+5x+3) = \lim_{x\to -1} (x^4-3x) \cdot \lim_{x\to -1} (x^2+5x+3)$

(Product Law)

$= \left(\lim_{x\to -1} x^4 - \lim_{x\to -1} 3x\right) \cdot \left(\lim_{x\to -1} x^2 + \lim_{x\to -1} 5x + \lim_{x\to -1} 3\right)$

(Sum & Difference Law)

$= \left(\left(\lim_{x\to -1} x\right)^4 - 3\lim_{x\to -1} x\right)\left[\left(\lim_{x\to -1} x\right)^2 + 5\lim_{x\to -1} x + \lim_{x\to -1} 3\right]$

(Power & Constant Multiple

$= [(-1)^4 - 3(-1)][(-1)^2 + 5(-1) + 3]$ (Law 7 & 8)

$= (1+3)(1-5+3)$

$= 4(-4+3)$

$= 4(-1)$

$= \boxed{-4}$

3. (a) As limits,

$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}}$

$= \lim_{x\to 2} x+3$

But when we can plug in 2,

$\frac{x^2+x-6}{x-2}$ gives a denominator of 0,

so DNE, but $x+3$ gives the value 5. They are equal when $x \neq 2$.

(b) $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$ CAN be simplified to

$\lim_{x\to 2} x+3$ since $x \to 2$ means "x IS NEAR

but NOT EQUAL TO 2". So,

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}} = \lim_{x\to 2} x+3$$

4. $$\lim_{x\to -3} \frac{x^2+3x}{x^2-x-12} = \lim_{x\to -3} \frac{x(x+3)}{(x+3)(x-4)}$$

$$= \lim_{x\to -3} \frac{x}{x-4}$$

$$= \frac{-3}{-3-4} \quad \text{(by Direct Substitution Property)}$$

$$= \frac{-3}{-7}$$

$$= \boxed{3/7}$$

5. $$\lim_{u\to 2} \frac{\sqrt{4u+1}-3}{u-2} = \lim_{u\to 2} \frac{\sqrt{4u+1}-3}{u-2} \cdot \frac{\sqrt{4u+1}+3}{\sqrt{4u+1}+3}$$

$$= \lim_{u\to 2} \frac{4u+1-9}{(u-2)(\sqrt{4u+1}+3)}$$

$$= \lim_{u\to 2} \frac{4u-8}{(u-2)(\sqrt{4u+1}+3)}$$

$$= \lim_{u\to 2} \frac{4\cancel{(u-2)}}{\cancel{(u-2)}(\sqrt{4u+1}+3)}$$

$$= \lim_{u\to 2} \frac{4}{\sqrt{4u+1}+3}$$

$$= \frac{\lim_{u\to 2} 4}{\lim_{u\to 2} \sqrt{4u+1}+3}$$

$$= \frac{\lim_{u\to 2} 4}{\lim_{u\to 2}\sqrt{4u+1} + \lim_{u\to 2} 3}$$

$$= \frac{\lim_{u\to 2} 4}{\sqrt{\lim_{u\to 2} 4u+1} + \lim_{u\to 2} 3}$$

$$= \frac{4}{\sqrt{8+1}+3}$$

$$= \frac{4}{\sqrt{9}+3}$$

$$= \frac{4}{3+3}$$

$$= \frac{4}{6}$$

$$= \boxed{2/3}$$

6. $$\lim_{h\to 0} \frac{(3+h)^{-1} - 3^{-1}}{h} = \lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{3(3+h)} - \frac{(3+h)}{3(3+h)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3-(3+h)}{3+h}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3-3-h}{3+h}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{-h}{3+h}}{h}$$

$$= \lim_{h\to 0} \frac{-h}{3+h} \div h$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{3+h} \cdot \frac{1}{\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-1}{3+h}$$

$$= \frac{-1}{3+0}$$

$$= \boxed{-1/3}$$

7. If $2x \le g(x) \le x^4 - x^2 + 2$ for all x, evaluate $\lim_{x\to 1} g(x)$.

$$\lim_{x\to 1} 2x \le \lim_{x\to 1} g(x) \le \lim_{x\to 1} x^4 - x^2 + 2$$

$$2 \le \lim_{x\to 1} g(x) \le 1^4 - 1^2 + 2$$

$$2 \le \lim_{x\to 1} g(x) \le 2$$

By the Squeeze Theorem: $\boxed{\lim_{x\to 1} g(x) = 2}$

8. $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right) = \lim_{x\to 0^+} \left(\frac{1}{x} \frac{-1}{x}\right)$ (Since $x > 0$ if $x \to 0^+$)

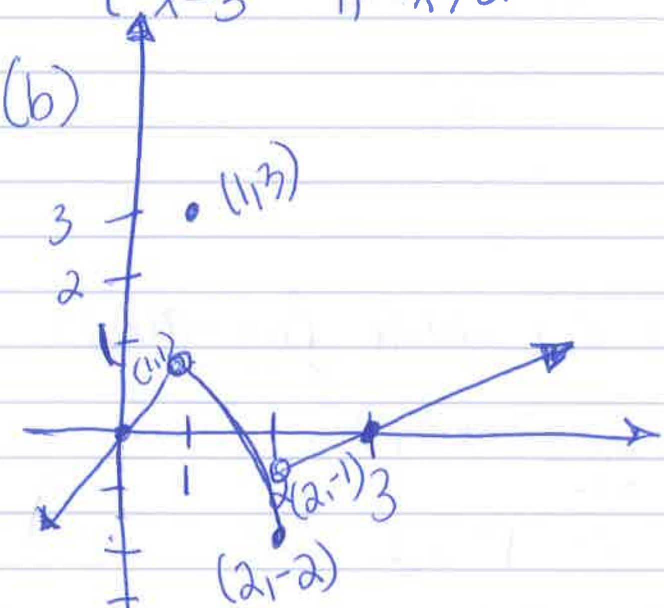
$$= \lim_{x\to 0^+} 0$$

$$= \boxed{0}$$

Note: I could <u>NOT</u> use the sum or difference Rules since $\lim_{x\to 0^+} \frac{1}{x}$ dne on its own.

9. $g(x) = \begin{cases} x & \text{if } x<1 \\ 3 & \text{if } x=1 \\ 2-x^2 & \text{if } 1<x\leq 2 \\ x-3 & \text{if } x>2 \end{cases}$

(b)



(a) (i) $\lim\limits_{x \to +1^-} g(x) = \boxed{1}$ (ii) $\lim\limits_{x \to 1} g(x) \neq \boxed{1}$ (iii) $g(1) = \boxed{3}$

(iv) $\lim\limits_{x \to 2^-} g(x) = \boxed{-2}$ (v) $\lim\limits_{x \to 2^+} g(x) = \boxed{1}$ (vi) $\lim\limits_{x \to 2} g(x)\ \boxed{DNE}$

10. $\lim\limits_{v \to c^-} L = \lim\limits_{v \to c^-} L_0 \sqrt{1 - \frac{v^2}{c^2}}$

$= L_0 \lim\limits_{v \to c^-} \sqrt{1 - \frac{v^2}{c^2}}$

$= L_0 \sqrt{\lim\limits_{v \to c^-} 1 - \frac{v^2}{c^2}}$

$= L_0 \sqrt{1 - \frac{c^2}{c^2}}$

$= L_0 \sqrt{1 - 1}$

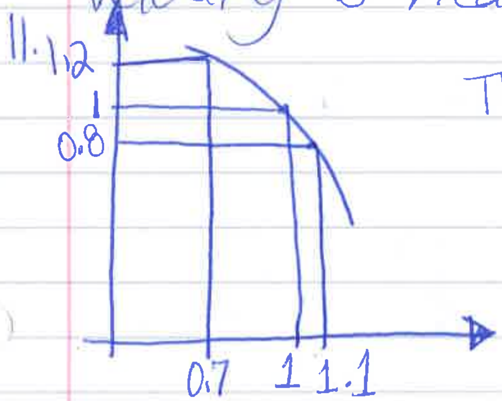
$= L_0 (0)$

$= \boxed{0}$

p.6

If we were to take the right-hand limit, we would be plugging in numbers <u>larger</u> than c, so $1-\frac{v^2}{c^2}$ would be <u>negative</u> & you cannot have a negative under a square root.

The result we get when $v \to c^-$ is what happens to the length of an object when its velocity is near-close to the speed of light.



The distance between 1 & 0.7 is:

$|0.7-1| = |-0.3| = 0.3$

The distance between 1 & 1.1 is:

$|1.1-1| = |0.1| = 0.1$

Every x-value in the interval (0.7, 1.1) is sent by f(x) to an interval (0.8, 1.2) (i.e. a distance of 0.2 away from f(1)). So δ has two options:

$\delta = 0.3$ or $\delta = 0.1$

If $\delta = 0.3$, our x-interval would be (0.7, 1.3) but there are values (say, 1.2) that does NOT go to (0.8, 1.2), so $\delta \neq 0.3$

If $\delta = 0.1$, our x-interval would be (0.9, 1.1) ~~but also~~ all values within this interval has f-values between (0.8, 1.2) so this δ works.

$\boxed{\delta = 0.1}$

The distance between 3 & 2.6 is:

$|2.6-3| = |-0.4| = 0.4$

The distance between 3 & 3.8 is:

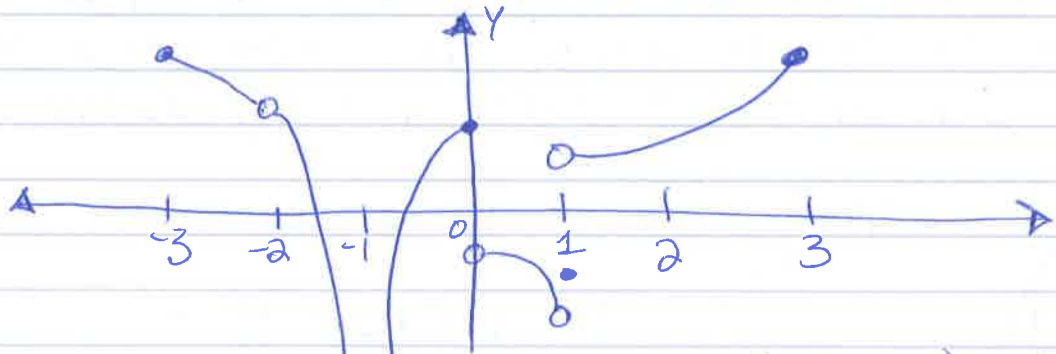
$|3.8-3| = |0.8| = 0.8$

For the same reason as above in #11, δ is the smaller of those two values, so $\boxed{\delta = 0.4}$

13. $\boxed{\lim_{x \to 4} f(x) = f(4)}$

14. It has no holes, jumps, or vertical asymptotes

15.



First I check any value in $(-3, -2)$. From the graph we see ① $\lim_{x \to a} f(x)$ exists ② $f(a)$ exists & ③ $\lim_{x \to a} f(x) = f(a)$ so on $(-2, 3)$, f is continuous.

At -2: We see $\lim_{x \to -2} f(x)$ exists, but $f(-2)$ DNE so -2 is a discontinuity.

From $(-2, -1)$ we see ① $\lim_{x \to a} f(x)$ exists ② $f(a)$ exists ③ $\lim_{x \to a} f(x) = f(a)$ so on $(-2, -1)$ f is continuous

At -1: We see $\lim_{x \to -1} f(x)$ exists But $f(-1)$ DNE so -1 is a discontinuity

From $(-1, 0)$: All 3 conditions are held for similar reasons to above so f is continuous on $(-1, 0)$

At 1: $\lim_{x \to 1} f(x)$ DNE so f has a discontinuity at 1

At 0: $\lim_{x\to 0} f(x)$ DNE so f has a discontinuity at 0.

From $(0,1)$: Same as before. f is continuous on $(0,1)$

From $(1,3)$: Same as before. f is continuous on $(1,3)$

All that's left to check is at 3 & -3

For continuity I only need to check continuity from the right or from the left.

At 3: ① $\lim_{x\to 3^-} f(x) = f(3)$ ② $f(3)$ exists ③ $\lim_{x\to 3^-} f(x)$ exists,

so f is continuous at 3 (from the left)

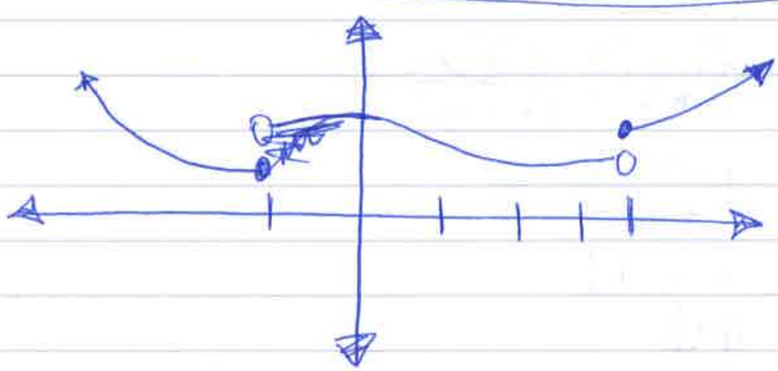
At -3: ① $\lim_{x\to -3^+} f(x) = f(-3)$ ② $f(-3)$ exists ③ $\lim_{x\to -3^+} f(x)$ exists

So $f(x)$ is continuous from the right at -3.

Putting it all together, we get f is continuous on

$$\boxed{[-3,-2)\cup(-2,-1)\cup(-1,0)\cup(0,1)\cup(1,3]}$$

16.



17. $\lim_{a\to 2} g(t) = \lim_{a\to 2} \frac{t^2+5t}{2t+1} = \lim_{a\to 2} \frac{t(t+5)}{2t+1}$

$= \frac{\lim_{a\to 2} t(t+5)}{\lim_{a\to 2} 2t+1} = \frac{\lim_{a\to 2} t \lim_{a\to 2} t+5}{\lim_{a\to 2} 2t+1}$

$= \frac{2\cdot 7}{5} = \frac{14}{5}$

$$g(2) = \frac{2^2+5\cdot 2}{2\cdot 2+1} = \frac{4+10}{4+1} = \frac{14}{5}$$

So $\lim_{a \to 2} g(t) = g(2)$ ✓

18. $f(x) = \begin{cases} \frac{2x^2-5x-3}{x-3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ $\qquad a=3$

$$x = \frac{+5 \pm \sqrt{(-5)^2-4(2)(-3)}}{2\cdot 2} = \frac{5 \pm \sqrt{25+24}}{4}$$
$$= \frac{5 \pm \sqrt{49}}{4}$$
$$= \frac{5 \pm 7}{4}$$

$x = \frac{12}{4}$ or $x = \frac{-2}{4}$

$x = 3$ or $x = -1/2$

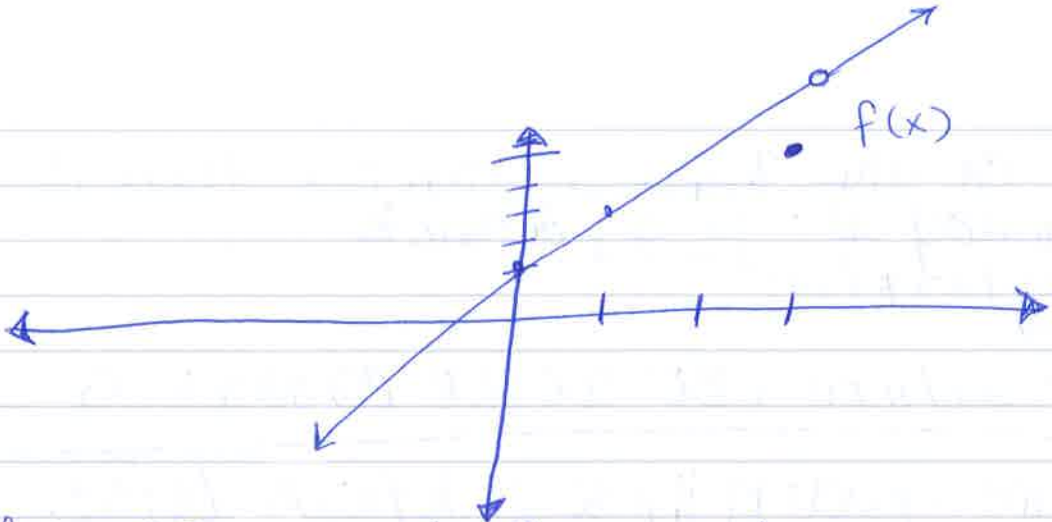
$$2x^2-5x-3 = (x-3)(2x+1)$$

$$\lim_{x \to 3} \frac{2x^2-5x-3}{x-3} = \lim_{x \to 3} \frac{(x-3)(2x+1)}{(x-3)}$$
$$= \lim_{x \to 3} 2x+1$$
$$= 2(3)+1$$
$$= 6+1$$
$$= 7$$

So $\lim_{x \to 3} f(x) = 7$ but $f(3) = 6$, so $\lim_{x \to 3} f(x) \neq f(3)$.

So f is discontinuous at $a=3$

$$\frac{2x^2-5x-3}{x-3} = \frac{(x-3)(2x+1)}{(x-3)} = 2x+1 \quad \underline{\text{if}} \quad x \neq 3$$



19. $f(x) = \dfrac{x^3-8}{x^2-4} = \dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$

$= \dfrac{x^2+2x+4}{x+2}$ if $x \neq 2$

$\lim\limits_{x \to 2} \dfrac{x^3-8}{x^2-4} = \lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x+2)}$

$= \lim\limits_{x \to 2} \dfrac{x^2+2x+4}{x+2}$

$= \dfrac{2^2+2\cdot 2+4}{2+2}$

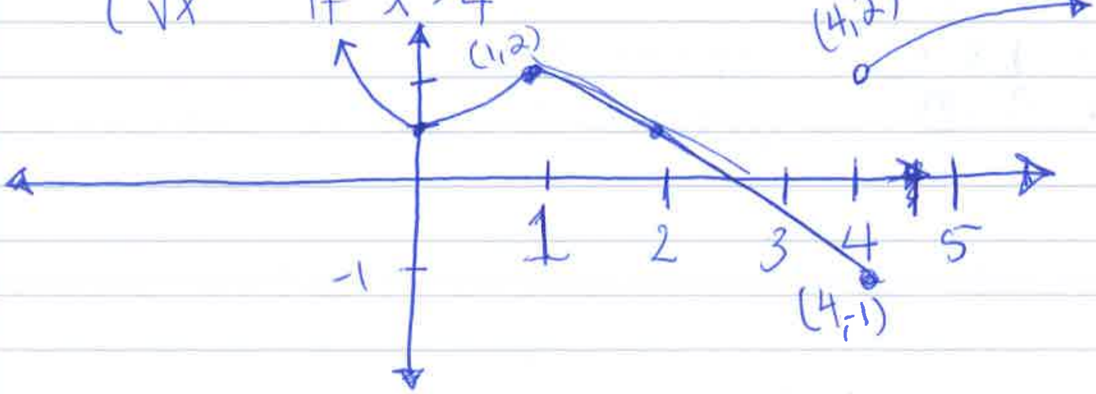
$= \dfrac{4+4+4}{4}$

$= \dfrac{12}{4}$

$= 3$

so $\boxed{f(2)=3}$

20. $f(x) = \begin{cases} x^2+1 & \text{if } x \leq 1 \\ 3-x & \text{if } 1 < x \leq 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$



P11

Looking at the graph we can see there is a discontinuity at $\boxed{x=4}$ because
$\lim_{x\to 4} f(x) \neq f(4)$

But everywhere else we are continuous

* If you are reading these solutions before the final or Exam 2, Thank you! ~~Homework~~ It is important to understand Homework solutions to suceed in Calculus. As a reward for reading these solutions, send me an email with subject "I read HW 3 solutions" & I'll give you 2 points back on HW3. *

Back to the problem.
now we look if f is continuous from the right or left.

$\lim_{x\to 4^-} f(x) = -1 = f(4)$

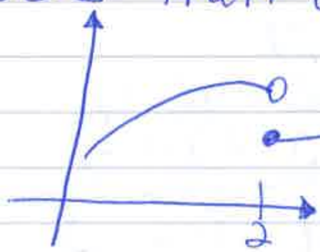
so $\boxed{\text{f is continuous from the left at } a=4}$

$\lim_{x\to 4^+} f(x) = 2 \neq f(4)$

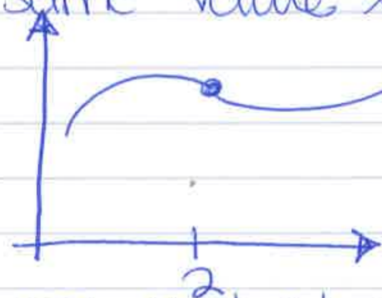
so f is discontinuous from the right at $a=4$

21. $f(x)=\begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x<2 \\ ax^2-bx+3 & \text{if } 2\le x<3 \\ 2x-a+b & \text{if } x\ge 3 \end{cases}$

In order for a piecewise function to be continuous then we can't have the following happen:



We need that when we switch from one function to another, the endpoints lay at the same value like so



So we need that at $x=2$:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} ax^2-bx+3$$

$$\lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = 4a-2b+3$$

$$\lim_{x\to 2} (x+2) = 4a-2b+3$$

$$4 = 4a-2b+3$$

$$1 = 4a-2b$$

We want to find a & b, but ONLY have 1 equation. With two unknowns we need two equations.

Similarly at $x=3$ we have:

$$\lim_{x\to 3} ax^2-bx+3 = \lim_{x\to 3} 2x-a+b$$

$$9a-3b+3 = 6-a+b$$

$$10a-4b = 3$$

Now we have 2 variables & 2 equations.

$$\begin{cases} 4a-2b=1 \\ 10a-4b=3 \end{cases}$$

Now I solve this linear system:

$$\begin{array}{l} 2\times(4a-2b=1) \\ \underline{10a-4b=3} \end{array} = \begin{array}{l} -8a-4b=1 \\ \underline{10a-4b=3} \\ -2a+0=-2 \\ -2a=-2 \\ \boxed{a=1} \end{array}$$

Now I plug in $a=1$ to find b:

$$10(1)-4b=3$$
$$10-4b=3$$
$$-4b=-7$$
$$\boxed{b=\frac{7}{4}}$$

22. $y=x^3-3x+1 \quad (2,3)$

$$f'(2)=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{x^3-3x+1-(8-6+1)}{x-2}$$

$$=\lim_{x\to 2}\frac{x^3-3x+1-3}{x-2}=\lim_{x\to 2}\frac{x^3-3x-2}{x-2}$$

I need to factor this by polynomial long division

$$\begin{array}{r} x^2+2x+1 \\ x-2\,\overline{)\,x^3+0x^2-3x-2} \\ -\underline{(x^3-2x^2)} \qquad\quad \\ 2x^2-3x \qquad \\ -\underline{(2x^2-4x)} \qquad \\ x-2 \\ -\underline{(x-2)} \\ 0 \end{array}$$

So $\dfrac{x^3-3x-2}{x-2} = \dfrac{(x^2+2x+1)(x-2)}{(x-2)}$ if $x \neq 2$

Therefore,

$$\lim_{x\to 2} \frac{x^3-3x-2}{x-2} = \lim_{x \to 2} \frac{(x^2+2x+1)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= \lim_{x\to 2} x^2+2x+1$$

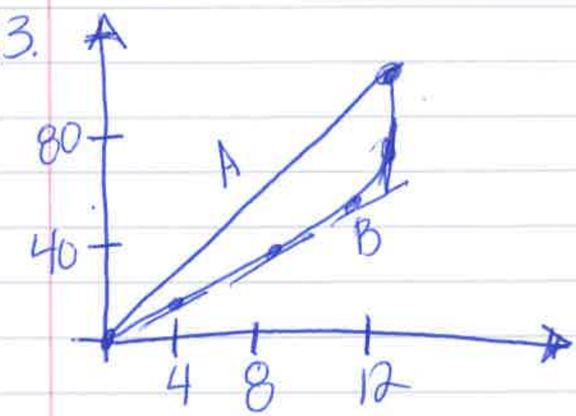
$$= 4+4+1$$

$$= 9$$

So the slope of our tangent line is 9

$\boxed{y-3 = 9(x-2)}$

23.



(a) A ran at a constant speed (velocity)
B ran & his velocity increased the entire time (i.e. B ran faster & faster)

(b) The distance between the runners was greatest at $\boxed{t=10}$

(c) They have the same velocity when the slope of the tangents are the same, around $\boxed{t=10}$