

$$-1 = 1 ?$$

Meijun Zu

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This article is mainly written for middle school students and their teachers; And for anyone who believes that "MATH IS FUN".

While learning operations on decimal numbers and polynomials, besides three basic operation rules for addition and multiplication (commutative law, associative law and distributive law), students also need to know the basic rules for exponential operations. For any two non-zero number a and b , and two integers m and n , we have

$$(1) a^m \times a^n = a^{m+n}, \quad (2) (a^n)^m = a^{nm} \quad \text{and} \quad (3) (ab)^m = a^m \times b^m.$$

In fact, from the definition of exponential operation, we can directly derive all three rules. Well, if you do not like the derivation, it is okay, but you must remember that **the above three rules only hold for m and n being integers**. Otherwise, if you think the above three rules hold for any exponents for m and n , you may get some weird conclusions.

Why? Well, if you think rule (2) holds for any exponents, then we have

$$[(-1)^{\frac{1}{2}}]^2 = [-1]^{\frac{1}{2} \times 2} = [-1]^1 = -1.$$

On the other hand,

$$[(-1)^{\frac{1}{2}}]^2 = [-1]^{\frac{1}{2} \times 2} \xleftrightarrow{\text{switch the order of the factors in the exponent}} [(-1)^{\frac{1}{2}}]^2 = [-1]^{2 \times \frac{1}{2}}.$$

Thus

$$[(-1)^{\frac{1}{2}}]^2 = [-1]^{\frac{1}{2} \times 2} = [-1]^{2 \times \frac{1}{2}} = [(-1)^2]^{\frac{1}{2}} = 1^{\frac{1}{2}} = 1.$$

You thus get a contradictory conclusion: $-1=1!$ Ridiculous, isn't it?

Well, if you think rule (3) holds for any exponents, then we have

$$[(-1)^{\frac{1}{2}}]^2 = (-1)^{\frac{1}{2}} \times (-1)^{\frac{1}{2}} = [(-1) \times (-1)]^{\frac{1}{2}} = 1^{\frac{1}{2}} = 1.$$

Again, we get a contradictory conclusion: $-1=1!$

When the exponent is a fraction, we have to introduce some new numbers (if the base numbers are positive, we will introduce irrational numbers; when the base numbers are negative, we then have to introduce imaginary numbers). Once we learn complex numbers, we need to modify the above three rules for exponential operators. Read Dr. Zhu's precalculus book for more details.