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Preface

0.1 About this book

The main goal of this book is to help college-bound students prepare for future challenges during their study of advanced mathematics.

In the first three chapters, we list the most important properties for number sets and the elementary functions, with the exception of trigonometric functions, which will be covered in Chapters 4 and 5.

In order to prepare students for college mathematics courses, we shall answer some questions the study of elementary mathematics may have left behind. Why do we have the Fundamental Assumption of Elementary Math? What on the earth, is a repeating decimal? And so on. We introduce some basic knowledge of limits, continuity, and series in Chapters 7 and Chapter 8. At the end of the book, we introduce derivatives and integrals for single variable functions. These concepts are often used in the study of other modern scientific subjects.

The corresponding mathematical level for each chapter is listed as follows:

Chapter 1. Numbers and Operations (Math Level: Algebra I students)

Chapter 2. Functions (Math Level: Algebra II students)

Chapter 3. Elementary Functions (Math Level: Algebra II students)

Chapter 4. Trigonometric Functions (Math Level: Precalculus students)

Chapter 5. Inverse Trigonometric Functions (Math Level: Precalculus students)

Chapter 6. Geometric properties of functions (Math Level: Precalculus students)

Chapter 7. Sequence and Series (Math Level: Calculus students)

Chapter 8. Differential and Integral (Math Level: Calculus students)

Chapter 9. AP Calculus Free-response Questions (Math Level: Calculus students)

I am indebted to everyone in my family for their immeasurable contributions to my lifelong education goal. I would like to thank my two children for helping me realize that the potential of students depends on trust. I would like to thank my wife for proofreading

the book and helping make our household conducive to learning. The cover of this book was designed by my friend Dr. Lei CAI.

0.2 Why do we need Calculus

Here we try to illustrate some motivations for studying differentiation and integration after we complete the study of "Introductory Algebra" and "Introductory Geometry and Proofs".

We first recall some old questions left to be answered.

1. Repeating (Recurring) decimals

Although we introduced the concept of a reciprocal of a nonzero integer, we did not show what a reciprocal number looks like. A naive arithmetic calculation leads us to the following confusing "long division":

$$
\begin{array}{cc|ccccc}
 & & 0 & .3 & 3 & 3 & 3 \\
\hline
3 &) & 1. & 0 & 0 & 0 & 0 \\
 & - & & 9 & & & \\
\cline{3-7}
 & & & 1 & 0 & & \\
 & & - & 0 & 9 & 0 & \\
\cline{3-7}
 & & & & 1 & 0 & \\
 & & & & - & 9 & \\
\cline{3-7}
 & & & & & & 1
\end{array}
$$

We thus have the following notation:

$$\frac{1}{3} = 0.\dot{3} = 0.33\cdots$$

and call the right-hand side a "repeating decimal number".

On closer inspection, the first step in the above calculation is correct:

$$\frac{1}{3} = 0.3 + \frac{0.1}{3}.$$

The next step is also okay:

$$\frac{1}{3} = 0.3 + 0.03 + \frac{0.01}{3}.$$

But who can guarantee that the computation can go on? Even if it can go on, what is the real meaning of this procedure? This is where the concept of an infinite series comes in.

2. Fundamental Assumption of Elementary Math

In Chapter 11 of "Introductory Algebra", we introduced the *Fundamental Assumption of Elementary Math*: *The operation rules that hold for all rational numbers are true for all real numbers.* Using this assumption we extend many rules holding for rational numbers to all real numbers. A typical example is the definition of the area of a rectangle. In this book, we will use continuity properties to verify that the Fundamental Assumption is correct.

3. Approximation by polynomials

In the study of Algebra and Geometry, we have already encountered some basic questions. For example, how is the value of $\sqrt{2.3}$ calculated? How is the value for π approximated?

Here is a related question: how does one find the tangent line for parabola $y = x^2$ at point $(1, 1)$? Essentially, we need to understand all elementary functions before using this understanding to comprehend more complicated functions.

Here is an analogy: if we want to know about an onion, we must peel off its outer skin layer by layer. Peeling off the outer skin is analogous to calculating the tangent line of a function. Once we know how to find the tangent line of a function at one point, we can then approximate the function near that point. This idea enables us to compute $\sqrt{2.3}$, π, and the value of e (via the Taylor series). It also helps us to better understand complex numbers. In particular, we will introduce how to derive the amazing Euler formula

$$e^{\mathrm{i}x} = \cos x + \mathrm{i} \sin x$$

4. Motivation from Physics

The study of motion also relies on calculus. The concept of derivatives enables us to understand instantaneous speed. Integration enables us to derive the distance formula from the speed function easily. In fact, differentiation and integration can also be used to derive Kepler's three famous theorems describing the motion of planets in our solar system.