Thoughts-on-Algebra-6

The first big obstacle in studying math ---fraction operations

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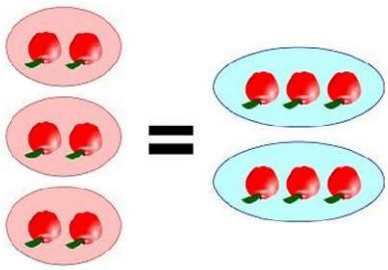
This article is suitable for students in third grade and above and their parents, primary school teachers, middle school teachers, young adults, and any mathematics educator who wants to learn/enjoy elementary mathematics again.



As shown in the picture above: horizontal addition: 3 of 2; vertical addition: 2 of 3 --- the commutative law of multiplication. Most children will enjoy this novel game when learning to count. If you don’t believe it, ask them: What is the sum of 100 of 3? Maybe some kids don't recite the multiplication table as quickly as you'd like, but the truth is: you'd be hard-pressed to find a few people in the world who don't remember the multiplication table. Sooner or later, children will memorize the 9×9 multiplication table. We also believe that by the third grade, few children will insist on adding 100 threes, one by one without understanding that they are actually three 100s, which is 300. With these in mind, we cannot fully understand why we do not teach our children that addition and multiplication satisfy the following three operation rules as soon as they learn the 9×9 multiplication table (around the second semester in the third grade):

For any three natural numbers a, b, c, we have

(a) $(Commutative\ rule)$ $\qquad a+b=b+a, \qquad a\times b=b\times a$。

(b) $(Associative\ rule)$ $\qquad a+b+c=(a+b)+c=a+(b+c),$

$$a \times b \times c = (a \times b) \times a = a \times (b \times c)。$$

(c) $(Distributive\ rule)$ $$a \times (b + c) = a \times b + a \times c。$$

One possibility is that Chinese children have not yet learned the English alphabet and cannot understand the formulas represented by letters. This guess does not seem to be true. American textbooks and the now famous Singaporean mathematics textbooks do not teach these three rules to children in a timely manner. Note that these three operation rules (which first hold true for natural numbers) hold true for new numbers introduced next (negative numbers, fractions, irrational numbers, and complex numbers). Not only that, these rules are also crucial to our understanding of newly introduced numbers.

We have also noticed that learning the definition and operations of fractions is the first big challenge that many children encounter in their journey of learning mathematics. The lack of understanding and mastery of the above operation rules may be one of the main reasons why these students find it difficult to learn fractions.

Almost all textbooks (except our books) have at least two very confusing points when introducing fractions and introducing operations on fractions. (1) There is no direct connection between fractions and integer division (or reciprocals). It takes students a long time to realize that m/n is actually m÷n, which is the same as m times 1/n. (2) Use the order of arithmetic operations for operations on fractions. First introduce the addition and subtraction of fractions, and then the multiplication and division of fractions. In fact, on the one hand, fractions already have multiplication and division operations, and it is emphasized that adding and subtracting first and then multiplying and dividing does not make logical sense; on the other hand, when introducing the addition of fractions with different denominators, equivalent fractions have to be introduced. As Professor Hung-Hsi Wu has long pointed out: the introduction of equivalent fractions without introducing the multiplication of fractions is flawed.

Equivalent fractions：for any two natural numbers m and n(here n is a nonzero natural number), and any positive natural number k,

$$\frac{m}{n} = \frac{m}{n} \times 1 = \frac{m}{n} \times \frac{k}{k} = \frac{mk}{mk}.$$

We use the product of two fractions in the last step.

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In our "Introductory Algebra" book, we eliminate the above confusing points. First , we define：

For any two natural numbers m and n (here n is a nonzero natural number)

(Definition 1) $$\frac{m}{n} = m \div n = m \times \frac{1}{n}.$$

Conventionally, we also call 1/n the reciprocal of a non-zero integer n. The above definition also indicates that dividing by a non-zero number is equal to multiplying by the reciprocal of this number.

For any two positive natural numbers *m and n* ，

（Formula 1） $$\frac{1}{n} \times \frac{1}{m} = \frac{1}{nm}.$$

(Formula 1) can be verified as follows: We only need to verify that the reciprocal of nm is 1/n×1/m. This is relatively easy to prove: we just need to use the commutative rule for multiplication, the associative rule and the definition of fractions (can you see where we use the commutative rule, where we use the associative rule, and where we use the definition of fractions in the following?).

Verifying (Formula 1)

$$nm \times \frac{1}{n} \times \frac{1}{m} = n \times \frac{1}{n} \times m \times \frac{1}{m} = 1.$$

From (Formula 1) and the definition of fractions (Definition 1), we can deduce the general fraction multiplication formula (try the derivation yourself)

For any two natural numbers k and l, and two positive natural numbers m and n， *(Formula 2)* $\frac{k}{n} \times \frac{l}{m} = \frac{kl}{nm}$.

After learning how to multiply fractions, let's look at adding fractions. First, for two fractions with the same denominator, we have

For any two natural numbers k and l, and one positive natural number n,

(Formula 3) $\frac{k}{n} + \frac{l}{n} = \frac{k+l}{n}$.

This can be verified by using the definition of fractions and the distributive rule (Formula 3).

Verifying (Formula 3)

$$\frac{k}{n} + \frac{l}{n} = k \times \frac{1}{n} + l \times \frac{1}{n} = (k + l) \times \frac{1}{n} = \frac{k+l}{n}.$$

For fractions with different denominators, we can use fraction multiplication to get two fractions with the same denominator, and then add them by using (Formula 3).

(Formula 4) $\frac{k}{n} + \frac{l}{m} = \frac{km}{nm} + \frac{nl}{nm} = \frac{km+nl}{nm}$.

How to simplify the results (simplify fractions) is another question (you need to use the decomposition properties of integers based on the Fundamental Theorem of Arithmetic), which will not be discussed here.