Chapter 9 Solutions to even numbered questions

Exercise 1.1.5 1(2). 131

1(4). 1254

1(6). −60

1(8). $= 48 \times \frac{1}{14} \times 7 = 24$

1(10). 18174

1(12). 11

1(14). $\frac{5}{21}$

1(16). $1\frac{4}{9}$

1(18). $\frac{1}{16}$

1(20). $1\frac{3}{4}$

2(2). 121

2(4).13200

2(6). $= 60 \times \frac{1}{3\times\frac{1}{2}} = 60 \times \frac{2}{3} = 40$

2(8). 96

2(10). 320

2(12). $\frac{23\times13+46}{1}5 + \frac{-13\times14+11\times7}{1}5 = \frac{23\times15}{1}5 + \frac{-15\times7}{15} = 16$

3(2). 7999

3(4). 8001

$$
\begin{array}{ccccc}
 & & 4 & -2 & 1 \\
\times & & & 2 & 1 \\
\hline
 & & 4 & -2 & 1 \\
+ & 8 & -4 & 2 & \\
\hline
 & 8 & 0 & 0 & 1.
\end{array}
$$

3(6). 421

$$\begin{array}{r|rrrr}
 & 4 & 2 & 1 & \\
\hline
2\hat{1} & 8 & 0 & 0 & \hat{1} \\
- & 8 & \hat{4} & & \\
\cline{2-5}
 & 0 & 4 & 0 & \\
 & -\,0 & 4 & \hat{2} & \\
\cline{2-5}
 & & & 2 & \hat{1} \\
 & & & -\,2 & \hat{1} \\
\cline{2-5}
 & & & & 0.
\end{array}$$

4(2). 324_5

4(4). 410_5

4(6). $= 215_5 \times 10_5 = 2150_5$

4(8). 10506_7

5(2). 2, 3, 5, 29

5(4). 7, 11, 13

5(6). 43, 47

6(2). 20

6(4). 12

7(2). $2\frac{1}{11}$

7(4). $\frac{13}{10}$

8(2). $2\frac{1}{7}$

8(4). 9

8(6). 2

9(2). 112_5

Exercise 1.2.3

1(2). $5x^2 - x$

1(4). $7x^2$

1(6). $3x^2 + 5$

1(8). $4x^2 - 6x - 2$

2(2). $x^4 + 3x^2 + 2$

2(4). $x^4 - 3x^2 + 2$

2(6). $x^4 - x^2 - 42$

2(8). $x^4 + x^2 - 42$

2(10). $2x^4 + 5x^2 - 3$

2(12). $2x^4 - 5x^2 - 3$

2(14). $x^6 - 1$

2(16). $x^6 + 1$

2(18). $x^6 + a^3$

3(2). $x^2 + 10x + 25$

3(4). $x^3 + 1$

3(6). $16x^2 + 4x + 1$

$$
\begin{array}{rrrrrr}
 & & & 16x^2 & +4x & +1 \\ \cline{3-6}
4x-1 &) & 64x^3 & 0 & 0 & -1 \\
 & - & 64x^3 & -16x^2 & & \\ \cline{3-6}
 & & 0 & 16x^2 & 0 & \\
 & - & 0 & 16x^2 & -4x & \\ \cline{3-6}
 & & & & 4x & -1 \\
 & & & - & 4x & -1 \\ \cline{3-6}
 & & & & & 0.
\end{array}
$$

3(8). $x^2 - x + 4$

$$
\begin{array}{rrrrrr}
 & & & 1x^2 & -x & +4 \\ \cline{3-6}
x+1 &) & x^3 & 0 & 3x & +4 \\
 & - & x^3 & +x^2 & & \\ \cline{3-6}
 & & 0 & -x^2 & +3x & \\
 & - & 0 & -x^2 & -x & \\ \cline{3-6}
 & & & & 4x & +4 \\
 & & & - & 4x & +4 \\ \cline{3-6}
 & & & & & 0.
\end{array}
$$

3(10). $4x^2 - 2x + 4$

4 (1). $\overline{1ab1ab} = \overline{1ab} \times 1001$. $1001 = 7 \times 11 \times 13$ is divisible by 7.

4(2).$\overline{1abcd1abcd} = \overline{1abcd} \times (10^5 + 1)$. $10^5 + 1$ is divisible by 11.

Exercise 1.3.5

1(2). $\frac{1}{5}$

1(4). $\frac{3}{13}$

1(6). $\frac{3}{5}$

1(8). $1\frac{7}{8}$

1(10). $10\frac{1}{8}$

1(12). $4 - \frac{3a}{2}$

1(14). Case 1: if $a = -1$, x can be any number; CAe 2: if $a \neq -1$, no solution.

1(16). Case 1: if $a \neq 1$, $x = \frac{b}{1-a}$; Case 2: if $a = -1$ and $b \neq 0$, no solution; case 3: if $a = -1$ and $b \neq 0$, x can be any number.

2(2). $x = 3, y = 2$

2(4). $x = \frac{5}{2}, y = \frac{4}{3}$

2(6). $x = -1\frac{8}{47},\ y = 1\frac{13}{47}$

3(2). $x = \frac{1}{3},\ y = \frac{1}{2}$

4(2). $x^2 + 1)(x^2 + 2)$

4(4).$(x - 1)(x + 1)(x^2 - 2)$

4(6). $(x^2 + 4)(x^2 - 6)$

4(8). $(2x^2 + 1)(x^2 + 3)$

4(10).$(2x - 1)(2x + 1)(x^2 + 3)$

4(12). $(x^2 + 2)(x^4 - 2x^2 + 4)$

4(14). $(x - 2)(x + 2)(x^2 + 4)$

4(16). $(x - 2)(x + 2)(x^2 - 2x + 4)(x^2 + 2x + 4)$

4(18). $(x^2 - x + 1)(x^2 + x + 1)$

5(2). $x = 2, or\ x = 3$

5(4). $x_1 = 2, x_2 = -2, x_3 = 3, x_4 = -3$

5(6). $x_1 = 2, x_2 = -2$

5(8). No rational solutions.

6(2). Ann's age is 5.

7(2). $a \neq 1$

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8(2). Case 1: $a \neq -1$, $x = \frac{b+1}{a+1}$; Case 2: $a = -1$ and $b \neq -1$, no solution; Case 3: $a = -1$ and $b = -1$, $x =$ any number.

Exercise 1.4.6

1(2). $x > -2$

1(4). $x > 1\frac{1}{8}$

1(6).$x > \frac{6}{a} + 1$

1(8). Case 1: $a > 0$, then $x < 2 + \frac{2}{a}$; Case 2: $a = 0$, $-2 < 0$, $x =$ any number; Case 3: $a < 0$, $x > 2 + \frac{2}{a}$.

1(10). Case 1: $a - 2b > 0$, then $x < \frac{2}{a+2b}$; Case 2: $a - 2b = 0$, $-2 < 0$, $x =$ any number; Case 3: $a - 2b < 0$, $x > \frac{2}{a+2b}$.

2(2). $-7 < x < 1$

2(4). $x < -7$ or $x > -3$

2(6). $x < 1$ or $x > 4$

3(2). Proof: $a^2 < b^2 \rightleftharpoons a^2 - b^2 < 0 \rightleftharpoons (a-b)(a+b) < 0$. Since $a + b > 0$, we know $(a-b)(a+b) < 0 \rightleftharpoons a - b < 0 \rightleftharpoons a < b$.

3(4). Proof: $x^2 > C^2 \rightleftharpoons (|x|)^2 - C^2 > 0 \rightleftharpoons (|x| - C)(|x| + C) > 0$. Since $|x| + C > 0$, we know $(|x| - C)(|x| + C) > 0 \rightleftharpoons |x| - C > 0 \rightleftharpoons |x| > C$.

3(6). Proof: First observe that

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz),$$

and

$$x^2 + y^2 + z^2 - xy - xz - yz = \frac{1}{2}[(x-y)^2 + (x-z)^2 + (y-z)^2] \geq 0.$$

For positive number x, y, z, $x + y + z > 0$. Thus $x^3 + y^3 + z^3 - 3xyz \geq 0$, which yields

$$x^3 + y^3 + z^3 \geq 3xyz.$$

Exercise 1.5.7

1(2). $5^{-\frac{1}{2}}$

1(4). x^3y

1(6). $3\sqrt{2} + \sqrt{3}$

1(8). $-\frac{\sqrt{2}}{2} - \frac{2\sqrt{3}}{3}$

1(10). 0

1(12). $\frac{1}{2}$

1(14). -1

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2(2). $-2\sqrt{7}-4\sqrt{5}$

2(4). $\sqrt{3}+2\sqrt{2}$

2(6). $2\frac{1}{2}$

3(2). Proof by contradiction. If $\sqrt{p}$ is a rational number, then

$$\sqrt{p}=\frac{m}{n},$$

where m, n are two positive integers and their largest common factor $(m, n)=1$ (we can assume $\frac{m}{n}$ is a simple fraction).

So we have $m^2=pn^2$, thus m is divisible by p. We write $m=pk$, where k is another positive integer. Bringing into the above equality and simplifying, we have

$$n^2=pk^2.$$

Thus n is divisible by p. So far, we know that both m, n are divisible by p, thus $(m, n)\geq p>1$. Contradicts to the assumption that $(m, n)=1$.

Exercise 1.6.4

1(2). $-1-5i$

1(4). 41

1(6). $5+4i$

1(8). $2i$

2(2). $|z|=1,\ arg(z)=\frac{5\pi}{3}$

2(4). $|z|=1,\ arg(z)=\frac{11\pi}{6}$

2(6). $|z|=1,\ arg(z)=\frac{7\pi}{4}$

2(8). $\sqrt{2}+\sqrt{2}i$

3(2). $x^2=(\sqrt{5}i)^2$. So $x_1=\sqrt{5}i,\ x_2=-\sqrt{5}i$

3(4). $x_1=\frac{-3+\sqrt{15}i}{2},\ x_2=\frac{-3-\sqrt{15}i}{2}$.

3(6). $x_1=e^{\frac{\pi}{6}}=\frac{\sqrt{3}}{2}+\frac{1}{2}i,\ x_2=e^{\frac{5\pi}{6}}=-\frac{\sqrt{3}}{2}+\frac{1}{2}i,\ x_1=e^{\frac{3\pi}{2}}=-i.$

4(2). $x^5=32$

Exercise 1.7.2

1(2). $1-\frac{\sqrt{2}}{4}$

1(4). 31

1(6). b^2

1(8). $8+i$

1(10). $8 - i$

1(12). $2x - 1$

1(14). $(x+1)^8 = x^8 + C_8^1 x^7 + C_8^2 x^6 + \ldots + C_8^7 x + 1$

2(2a). For $a \neq 0$, $x = \frac{a}{2}$.

2(2b). For $a \neq -1$, $x = 2a + 2$; For $a = -1$, no solution.

2(4a). $x = 4,\ or\ x = 7$.

2(4b). $x = -\frac{3}{2},\ or\ x = \frac{1}{3}$.

2(6a). $x = 3$.

2(6b). $x = 6$.

2(8a). $-1 \leq x \leq 0$.

2(8a). $-2 \leq x < 0$.

3(2). Proof:

$$A > B \Rightarrow A + x > B + x \quad (\textbf{Addition invariant})$$

$$x > y \Rightarrow B + x > B + y \quad (\textbf{Addition invariant})$$

Since $A + x > B + x$ and $B + x > B + y$, using **transitive property** we conclude: $A + x > B + y$.

3(4). Proof: For any two real numbers x, y,

$$(x - y)^2 \geq 0 \Rightarrow x^2 + y2 \geq 2xy.$$

If x, y are positive, we have (since $f(x) = \ln x$ is an increasing function)

$$\ln(x^2 + y^2 \geq \ln(2xy) = \ln 2 + \ln x + \ln y.$$

This yields

$$\ln(x^2 + y^2) - \ln 2 \geq \ln x + \ln y.$$

Exercise 2.1.4

2(1). $A \cap B = \{x \in \mathbb{R} \mid 2 < x < 12\}$.

2(2). $A \cup B = \{x \in \mathbb{R} \mid x > -8\}$

2(3). $A^c \cup B^c = \{x \in \mathbb{R} \mid x < 2,\ orx > 12\}$.

2(4). Since $(A \cap B)^c = \{x \in \mathbb{R} \mid x < 2,\ orx > 12\}$. The proof for general sets is given for Proposition 2.2.

Exercise 2.2.5

1(2). (a) 51. (b) 33. (c) 16. (d). $51 - 16 = 35$. (e). $33 - 16 = 17$.

2(1). 10. $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{A, D\}$, $\{A, E\}$, $\{B, C\}$, $\{B, D\}$, $\{B, E\}$, $\{C, D\}$, $\{C, E\}$, $\{D, E\}$

2(2). $C_5^2 \cdot C_3^2/2 = 15$ pairs.

$\{A, B\} and\ \{C, D\}$, $\{A, B\} and\ \{C, E\}$, $\{A, B\} and\ \{D, E\}$;
$\{A, C\} and\ \{B, D\}$, $\{A, C\} and\ \{B, E\}$, $\{A, C\} and\ \{D, E\}$;
$\{A, D\} and\ \{B, C\}$, $\{A, D\} and\ \{B, E\}$, $\{A, D\} and\ \{C, E\}$;
$\{A, E\} and\ \{B, C\}$, $\{A, E\} and\ \{B, D\}$, $\{A, E\} and\ \{C, D\}$;
$\{B, C\} and\ \{D, E\}$, $\{B, D\} and\ \{C, E\}$, $\{B, E\} and\ \{C, D\}$;

Exercise 2.3.7

2(1). (ii): $D = \{x \in \mathbb{R} \mid x \neq 1\}$, $R = \{y \in \mathbb{R} \mid y \neq 1\}$.
(iv): $D = \{x \in \mathbb{R} \mid x \geq \sqrt{3}\ or\ x \leq -\sqrt{3}\}$, $R = \{y \in \mathbb{R} \mid y \geq 0\}$.
2(2). (i): $f[g(x)] = e^{x^2-1}$; (ii): $g[f(x)] = e^{2x} - 1$.

3(2). Proof: $g(x)$ is even, so $g(-x) = g(x)$. Thus

$$f[g(-x)] = f[g(x)].$$

That is: $y = f[g(x)]$ is an even function.

3(4). Proof: Since $f(x)$ and $g(x)$ are increasing functions on $\mathbb{R}$, we know that for any $x_1 < x_2$,

$$f(x_1) \leq f(x_2), \quad \text{and} \quad g(x_1) \leq g(x_2).$$

So,

$$f(x_1) + g(x_1) \leq f(x_2) + g(x_2).$$

Thus $y = f(x) + g(x)$ is an increasing function.

$y = f(x)g(x)$ may not be an increasing function. For example: $f(x) = x$ and $g(x) = x - 1$ are increasing. But $y = x^2 - x$ is not an increasing function.

3(6). Proof: Since $y = g(x)$ is a periodic function, there is a $T > 0$, such that $g(x + T) = g(x)$. So

$$f[g(x + T)] = f[g(x)].$$

Thus $y = f[g(x)]$ is also a periodic function with T as its period.

Exercise 2.4.2

1(2). $A \cap B = B = \{6n \mid n \in \mathbb{N}\}$, $A \cup B = A = \{2n \mid n \in \mathbb{N}\}$, $A \cap (B \cup C) = A = \{2n \mid n \in \mathbb{N}\}$.

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1(4). 4.

2(2). $D = \{x \in \mathbb{R} \mid 1 \le x \le 3\}$, $R = \{y \in \mathbb{R} \mid y \ge 0\}$.

2(4). $D = \{x \in \mathbb{R} \mid x \neq -3\}$, $R = \{y \in \mathbb{R} \mid y \neq 1\}$.

2(6). $D = \{x \in \mathbb{R} \mid x \neq 1,\ x \neq 3\}$, $R = \{y \in \mathbb{R} \mid y < 0\ \ or\ y \ge \frac{1}{7}\}$.

2(8). (i). $g \circ g \circ g(x) = \frac{1}{8}x + 3\frac{1}{2}$; (ii). $g^{-1}(x) = 2x - 2$; (iii). $R(f) = \{y \in \mathbb{R} \mid y \ge -3$; (iv). $f^{-1}(-3) = -2$.

3(2). If $f(x)$ is an even function, then for $x_0 < 0$,

$$f(-x_0) = f(x_0).$$

Since $-x_0 < x_0$, the above inequality contradicts to the fact that $f(x)$ is an increasing function.

3(4). Since $f(0) = [f(0)]^2$ and $f(0) \neq 0$, we know that $f(0) = 1$. For all natural number b,

$$1 = f(0) = f(0)f(b) = f(b).$$

So $f(2022) = 1$.

Exercise 3.1.5

1(2). $x = \frac{3+\sqrt{5}}{4}, \quad or\ x = \frac{3-\sqrt{5}}{4}$.

1(4). $m < -\frac{9}{2}$.

1(6). $x > 3$ or $x < -3$.

1(8). $-1 < x < 3$.

2(2). $x = -4$, or $x = 4$, or $x = 5$.

2(4). $C_{10}^{8} = 45$.

Exercise 3.2.3

1(2). (i). $D = \{x \in \mathbb{R} \mid x \ge -4\}$. $R = \{y \in \mathbb{R} \mid y \ge 0\}$. $f^{(}x) = x^2 - 4$.

1(2). (ii). $D = \mathbb{R}$. $R = \{y \in \mathbb{R} \mid y \ge 3\}$. $f^{(}x) = x^2 - 4$. $f^{-1}(x) = \frac{2+\sqrt{4y-12}}{2}$ or $f^{-1}(x) = \frac{2-\sqrt{4y-12}}{2}$.

2(2). $x = -59$.

2(4). $x = 7$ ($x = 4$ is an extraneous solution).

Exercise 3.3.1

2(2). $D = \mathbb{R}$, $R = \{y \in \mathbb{R} \mid y \ge 2\}$.

3(2). Proof. Step 1, we prove, via the induction that, for any positive number $0 < p < 1$ and positive integer n,

$$(1-p)^n \geq 1 - pn.$$

For $n = 1$, $1 - p = 1 - p$. The statement is true.

Assume that the inequality holds for $n = k$. That is, for $n = k$,

$$(1-p)^k \geq 1 - pk.$$

Then, for $n = k + 1$, using induction assumption, we have

$$\begin{aligned}(1-p)^{k+1} &\geq (1-pk)(1-p)\\ &= 1 - p(k+1) + p^2 k\\ &\geq 1 - p(k+1).\end{aligned}$$

We thus prove the statement.

Step 2, we show that for all positive integers n,

$$(1+\frac{1}{n})^n < (1+\frac{1}{n+1})^{n+1}.$$

First we observe

$$\begin{aligned}\frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^n}\\ &= \frac{(\frac{n+2}{n+1})^{n+1}}{(\frac{n+1}{n})^n}\\ &= (\frac{(n+2)n}{(n+1)^2})^n \cdot \frac{n+2}{n+1}\\ &= \left(1 - \frac{1}{(n+1)^2}\right)^n \cdot (1 + \frac{1}{n+1})\\ &\geq \left(1 - \frac{n}{(n+1)^2}\right) \cdot \left(1 + \frac{1}{n+1}\right) \quad \text{(Using the inequality obtained in step 1)}\\ &= \left(1 - \frac{1}{n+1} + \frac{1}{(n+1)^2}\right) \cdot \left(1 + \frac{1}{n+1}\right)\\ &= 1 + \frac{1}{(n+1)^3}\\ &> 1.\end{aligned}$$

Step 3. So we have, for all positive integers n,

$$2 = (1+1)^1 \leq (1+\frac{1}{n})^n$$

which yields

$$2^{\frac{1}{n}} \leq 1 + \frac{1}{n}.$$

Exercise 3.4.1

2(2). (a). $D = \mathbb{R}$, $R = \{\in \mathbb{R} \mid y \geq -2\}$; (b). $D = \{x \in \mathbb{R} \mid x \geq \sqrt{3}\ or\ x \leq -\sqrt{3}\}$, $R = \{y \in \mathbb{R} \mid y \geq 0\}$.

2(4). (a). $f[g(x)] = xe^x$; (b). $g[f(x)] = x^x$.

3(2). (a). $x = 1 + \sqrt{2}$; (b). $x = 1 + \sqrt{2}$ or $x = 1 - \sqrt{2}$.

3(4). (a). $x = 4$; (b). $x = 4$ or $x = -2$.

3(6). Proof. Step 1, we prove that for three positive numbers a, b, c,

$$a^3 + b^3 + c^3 \geq 3abc.$$

In fact

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) \\ &= \frac{1}{2}(a + b + c)[(a - b)^2 + (a - c)^2 + (b - c)^2] \\ &\geq 0. \end{aligned}$$

Step 2. Let $x = a^3,\ y = b^3,\ z = c^3$. We obtain from step 1 that

$$x + y + z \geq 3(xyz)^{\frac{1}{3}}.$$

Thus

$$\frac{x + y + z}{3} \geq (xyz)^{\frac{1}{3}}.$$

Since $f(x) = \ln x$ is an increasing function, we have

$$\ln(\frac{x + y + z}{3} \geq \ln(xyz)^{\frac{1}{3}} = \frac{1}{3}\ln(xyz) = \frac{\ln x + y + \ln z}{3}.$$

Exercise 3.5.1

1(2).

(2.2). $D = \mathbb{R},\ R = \{y \in \mathbb{R} \mid y \geq 2^{-\frac{1}{4}}\}$.

1(2).

(2.4). $D = \mathbb{R},\ R = \{y \in \mathbb{R} \mid y > 0\}$.

1(3).

(3.2). $x^3 + 3x^2 - 2x + 1$; (3.4). $\frac{2}{3}$; (3.6). 2.

2(1).

(1.2). $x = \frac{5 \pm \sqrt{17}}{2}$; (1.4). $x_1 = x_2 = 1,\ x_3 = 2$; (1.6). $x = 0\ or\ x = 3$; (1.8). $x = 1,\ or\ x = 3$; (1.10). $x = \frac{-3 + 3\sqrt{17}}{2}$.

2(2).

(2.2). $x \geq \sqrt{2}\ or\ x < -\sqrt{5}$; (2.4). $x \geq 2$; (2.6). $x > 3\ or\ x < 1$.

3(2). $x = \frac{11 \pm 3\sqrt{17}}{8}$.

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3(4). $m < 0$ *or* $m > 12$.

Exercise 4.1.1

1(2). $-\frac{1}{2}$; 1(4). $\frac{1}{2}$; 1(6). $-\frac{\sqrt{3}}{3}$.

2(1). (ii). $T = \frac{8\pi}{3}$; (iv). $T = 2\pi$.

2(2). (ii). It is an even function; (iv). It is an even function.

3(3). $-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$.

Exercise 4.2.8

1(2). $\frac{1}{\sin\theta} = \csc\theta$.

1(4). $\cos^2 x$.

1(6). 1

2(2). Proof:

$$\frac{1}{1-\cos^2 x} = \frac{1}{\sin^2 x}.$$

and

$$\begin{aligned} 1+\cot^2 x &= 1+\frac{\cos^2 x}{\sin^2 x} \\ &= \frac{\sin^2 x+\cos^2 x}{\sin^2 x} \\ &= \frac{1}{\sin^2 x}. \end{aligned}$$

So, $\frac{1}{1-\cos^2 x} = 1+\cot^2 x$.

2(4). Proof.

$$\begin{aligned} \tan(\frac{\pi}{4}+x) &= \frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\cdot\tan x} \\ &= \frac{1+\tan x}{1-\tan x}. \end{aligned}$$

2(6). Proof.

$$\begin{aligned} \frac{\cos(x-y)}{\sin x\cos y} &= \frac{\cos x\cos y+\sin x\sin y}{\sin x\cos y} \\ &= \frac{\cos x}{\sin x}+\frac{\sin y}{\cos y} \\ &= \cot x+\tan y. \end{aligned}$$

2(8). Proof.

$$\begin{aligned}\sin 3\alpha &= \sin 2\alpha\cos\alpha + \cos 2\alpha \sin\alpha\\ &= 2\sin\alpha\cos^2\alpha + (1-2\sin^2\alpha)\sin\alpha\\ &= 2\sin\alpha - 2\sin^3\alpha + \sin\alpha - 2\sin^3\alpha\\ &= -4\sin^3\alpha + 3\sin\alpha.\end{aligned}$$

2(10). Proof.

$$\begin{aligned}\cos^4 5\alpha - \sin^4 5\alpha &= (\cos^2 5\alpha - \sin^2 5\alpha)(\cos^2 5\alpha + \sin^2 5\alpha)\\ &= \cos^2 5\alpha - \sin^2 5\alpha\\ &= \cos 10\alpha.\end{aligned}$$

2(12). Proof.

$$\begin{aligned}\frac{\sin 6x}{\sin 5x + \sin x} &= \frac{2\sin 3x\cos 3x}{2\sin 3x\cos 2x}\\ &= \frac{\cos 3x}{\cos 2x}.\end{aligned}$$

3(2). (i). $x = 1 \pm \sqrt{2}i$; (ii). $x_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $x_2 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, $x_3 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, $x_4 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $x_5 = 2$, $x_6 = -2$, $x_7 = 2i$, $x_8 = -2i$.

3(4). $y = 1 - \frac{1}{2}\sin^2 2x$, $R = \{y \in \mathbb{R} \mid \frac{1}{2} \le y \le 1\}$.

Exercise 4.3.1

1(2). $c = \sqrt{89 - 40\sqrt{3}}$.

1(4). $\sin(\angle A + \angle C) = \sin\angle B = \frac{4}{5}$.

2(2). $a = 10$.

2(4). Proof.

$$\begin{aligned}\frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} &= \tan\left(\frac{A}{2} + \frac{B}{2}\right)\\ &= \tan\left(\frac{\pi}{2} - \frac{C}{2}\right)\\ &= \frac{1}{\tan\frac{C}{2}}.\end{aligned}$$

This yields

$$\tan\frac{A}{2}\tan\frac{C}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{A}{2}\tan\frac{B}{2} = 1.$$

Exercise 4.5.1

1(2). Not even, not odd; 2π-periodic function.

1(4). Even function.

2(2). $D = \mathbb{R}$; $R = \{y \in \mathbb{R} \mid 1 \le y \le 5\}$ $(y = (\sin x + 1)^2 + 1)$

2(4). $D = \mathbb{R}$; $R = \{y \in \mathbb{R} \mid -3 \le y \le \frac{3}{2}\}$ $(y = (\frac{3}{2} - 2(\sin x - \frac{1}{2})^2)$

3(2). $r_1 = 2e^{\frac{\pi}{5}}$ (principal root), $r_2 = 2e^{\frac{3\pi}{5}}$, $r_3 = 2e^{\frac{5\pi}{5}} = -2$, $r_4 = 2e^{\frac{7\pi}{5}}$, $r_5 = 2e^{\frac{9\pi}{5}}$.

4(2). Proof.

$$\begin{aligned}(\cos x - \sin x)^2 &= \cos^2 x - 2\cos x \sin x + \sin^2 x \\ &= 1 - 2\cos x \sin x \\ &= 1 - \sin 2x.\end{aligned}$$

4(4). Proof. First, we note that $x_1 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$ is a solution to $x^5 + 1 = 0$. So (since $x \neq -1$)

$$x_1^4 - x_1^3 + x_1^2 - x_1 + 1 = 0.$$

Thus

$$\begin{aligned}0 &= Re\{x_1^4 - x_1^3 + x_1^2 - x_1 + 1\} \\ &= \cos\frac{4\pi}{5} - \cos\frac{3\pi}{5} + \cos\frac{2\pi}{5} - \cos\frac{\pi}{5} + 1 \\ &= -\cos\frac{\pi}{5} + \cos\frac{2\pi}{5} + \cos\frac{2\pi}{5} - \cos\frac{\pi}{5} + 1 \\ &= -2\cos\frac{\pi}{5} + 2\cos\frac{2\pi}{5} + 1.\end{aligned}$$

This yields

$$\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2}.$$

5(2). (a).

$$\begin{aligned}x^5 + x^4 + x^3 + x^2 + x + 1 &= \frac{x^6 - 1}{x - 1} \\ &= \frac{(x^3 - 1)(x^3 + 1)}{x - 1} \\ &= \frac{x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)}{x - 1} \\ &= (x + 1)(x^2 + x + 1)(x^2 - x + 1).\end{aligned}$$

5(2). (b). $x_1 = -1$, $x_2 = \frac{-1+\sqrt{3}i}{2}$, $x_3 = \frac{-1-\sqrt{3}i}{2}$, $x_4 = \frac{1+\sqrt{3}i}{2}$, $x_2 = \frac{1-\sqrt{3}i}{2}$.

6(2). $x = \frac{\pi}{3} + n\pi$ or $x = -\frac{\pi}{3} + n\pi$, where $n = 0, \pm 1, \pm 2, \cdots$.

6(4). $x = \frac{n\pi}{2}$, or $x = \frac{2\pi}{3} + 2n\pi$, or $x = \frac{4\pi}{3} + 2n\pi$where $n = 0, \pm 1, \pm 2, \cdots$.

Exercise 5.1.4

1(2). (a). $-\frac{\pi}{6}$; (b). $x = -\frac{\pi}{6} + n\pi$, where $n = 0, \pm 1, \pm 2, \cdots$.

1(4). (a). $\frac{3\sqrt{7}}{8}$; (b). $\frac{\pi}{9}$.

2(2). $\frac{4\sqrt{5}}{9}$.

2(4). $2x\sqrt{1-x^2}$.

Exercise 5.2.3

1(2). $\frac{\sqrt{15}}{60}$.

1(4). $\frac{3\sqrt{21}-1}{16}$.

2(2). $(\cos x + \sin x)(\cos x - \sin x - 1) = 0$. So, $x = \frac{3\pi}{4} + n\pi$, or $x = 2n\pi$, or $x = -\frac{\pi}{2} + 2n\pi$, where $n = 0, \pm 1, \pm 2, \cdots$.

2(4). $-1 \le x < 0$.

Exercise 5.3.1

1(2). Even in the domain $D = \{x \in \mathbb{R} \mid -\frac{\sqrt{2}}{2} \le x \le \frac{\sqrt{2}}{2}\}$.

1(4). Odd and strictly increasing function in the domain $D = \{x \in \mathbb{R} \mid -\infty x \le \infty\}$.

2(2). $-1 \le y \le 1$.

2(4). $\frac{1}{2} < x \le 1$.

Exercise 6.1.1

2(1). $(A, B) = 4$.

2(2). 2.

2(3). $2\sqrt{2}$.

Exercise 6.2.1

1(2). x-intercept: $x = \frac{7}{2}$; y-intercept: $y = -\frac{7}{5}$;

1(4). $(\frac{9}{7}, \frac{2}{7}, -\frac{15}{7})$.

2(2). Proof. Using Example 6.9, we have

$$\begin{aligned} \frac{a}{b} + \frac{b}{c} + \frac{c}{a} &\ge 3\sqrt[3]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}} \\ &= 3. \end{aligned}$$

2(4). Proof: Step 1: for $k = \frac{1}{2}$,

$$\begin{aligned} f(\frac{x_1+x_2}{2}) &= f(\frac{1}{2}x_1 + \frac{1}{2}x_2) \\ &\le \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2) \quad \text{(f(x) is a convex function)} \\ &= \frac{f(x_1)+f(x_2)}{2}. \end{aligned}$$

The statement is correct.

We now assume that for $n = k \ge 2$ the statement is correct.That is, for $n = k \ge 2$,

$$f(\frac{x_1+x_2+\cdots+x_k}{k} \le \frac{1}{k}f(x_1) + \frac{1}{2}f(x_2) + \cdots + f(x_k).$$

Step 2. Now for $n = k+1$,

$$\begin{aligned} & f(\frac{x_1+x_2+\cdots+x_{k+1}}{k+1}) \\ &= f(\frac{k}{k+1}\cdot(\frac{x_1+x_2+\cdots+x_k}{k}) + \frac{x_{k+1}}{k+1}) \\ &\le \frac{k}{k+1}f(\frac{x_1+x_2+\cdots+x_k}{k}) + \frac{1}{k+1}\cdot f(x_{k+1}) \quad \text{(since f(x) is a convex function)} \\ &\le \frac{1}{k+1}\big(f(x_1+f(x_2)+\cdots+f(x_k)\big) + \frac{1}{k+1}\cdot f(x_{k+1}) \quad \text{(using the induction assumption)} \\ &= \frac{f(x_1)+f(x_2)+\cdots+f(x_{k+1})}{k+1}. \end{aligned}$$

We prove that the statement holds for $n = k+1$ thus complete the proof.

Exercise 6.3.3

2(2). $x^2 + y^2 - 3y - 3\sqrt{x^2+y^2} = 0$.

Exercise 6.4.4

1(2). (a). 0; (b). 5; (c). $(0, 0, -12)$.

2(2). (a). Direct proof. Since

$$x + 16y + 64z \le |x| + 16|y| + 64|z|,$$

we only need to prove the inequality for non-negative numbers x, y, z. For non-negative numbers x, y, z,

$$\begin{aligned} & x + 16y + 64z \le 9\sqrt{x^2+16y^2+64z^2} \\ &\Leftrightarrow (x+16y+64z)^2 \le 81(x^2+16y^2+64z^2) \\ &\Leftrightarrow 0 \le 81(x^2+16y^2+64z^2) - (x+16y+64z)^2. \end{aligned}$$

Notice:

$$\begin{aligned}
81(x^2+16y^2+64z^2)-(x+16y+64z)^2 \\
&= 80x^2+65\cdot 16y^2+17\cdot 64z^2-16\cdot 2xy-64\cdot 2xz-16\cdot 64\cdot 2yz \\
&= 16(x-y)^2+64(x-z)^2+64\cdot 16(y-z)^2 \\
&\geq 0.
\end{aligned}$$

We thus complete the proof of the inequality.

(b). Let $\overline{u}=(1,4,8)$ and $\overline{v}=(x,4y,8z)$. Notice that

$$|\overline{u}|=\sqrt{1^2+4^2+8^2}=9, \quad \text{and} \quad |\overline{v}|=\sqrt{x^2+16y^2+64z^2}.$$

From Cauchy-Schwartz inequality, we know

$$\begin{aligned}
x+16y+64z &= \overline{u}\cdot\overline{v} \\
&\leq |\overline{u}||\overline{v}| = 9\sqrt{x^2+16y^2+64z^2}.
\end{aligned}$$

Exercise 6.5.4

1(2).

(i). $-3x-y+3z=0$.

(ii). $2x+3y+z=13$.

(iii). $2x+3y-z=\frac{5}{3}$.

2(2). They are not co-planar.

Exercise 6.6.3

2(2).

$$\begin{cases} x &= \frac{1}{2}(\sqrt{18-3t^2}-t), \quad -\sqrt{6}\leq t\leq\sqrt{6} \\ y &= \frac{1}{2}(-\sqrt{18-3t^2}-t), \quad -\sqrt{6}\leq t\leq\sqrt{6} \\ z &= t. \quad -\sqrt{6}\leq t\leq\sqrt{6} \end{cases}$$

It is a space circle.

Exercise 6.7.1

1(2). $B(\frac{1}{25},\frac{43}{25})$.

1(4). Choose a point $P_1(x_1,y_1)$ on the line. Thus $ax_1+by_1=c$. The unit normal vector of the line is: $\vec{n} = (\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}})$. So

$$\begin{aligned} d &= |\vec{P_0P_1} \cdot \vec{n}| \\ &= |\frac{a(x_1 - x_0) + b(y_1 - y_0)}{\sqrt{a^2 + b^2}}| \\ &= |\frac{c - ax_0 - by_0}{\sqrt{a^2 + b^2}}| \\ &= |\frac{ax_0 + by_0 - c}{\sqrt{a^2 + b^2}}|. \end{aligned}$$

2(2). $B(\frac{5}{9}, 2\frac{1}{9}, 1\frac{1}{9})$.

2(4). Choose a point $P_1(x_1, y_1, z_1)$ on the plane. Thus $ax_1 + by_1 + cz_1 = d$. The unit normal vector of the plane is: $\vec{n} = (\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}})$. So

$$\begin{aligned} d &= |\vec{P_0P_1} \cdot \vec{n}| \\ &= |\frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{\sqrt{a^2 + b^2 + c^2}}| \\ &= |\frac{d - ax_0 - by_0 - cz_0}{\sqrt{a^2 + b^2 + c^2}}| \\ &= |\frac{ax_0 + by_0 + cz_0 - d}{\sqrt{a^2 + b^2 + c^2}}|. \end{aligned}$$

4(2). $\vec{AB} \cdot \vec{AC} \times \vec{AD} = 0$. They are coplanar. The plane equation is: $-3x - 2y + 5z = 0$.

5(2). $y = \frac{5}{2}x^2 + \frac{1}{10}$.

6. (1). $\mathcal{L}_1$:

$$\begin{cases} x &= t, \quad -\infty < t < \infty \\ y &= 0 \\ z &= 1 \end{cases}$$

$\mathcal{L}_2$:

$$\begin{cases} x &= t, \quad -\infty < t < \infty \\ y &= 1 - t, \quad -\infty < t < \infty \\ z &= 0 \end{cases}$$

(2). Equation for plane $\mathcal{P}$: $z = 0$.

(3). $d = 1$

Exercise 7.1.6

1(2). $a_n = 3 \cdot 5^{-n+1}$.

1(4). 5.55555

2. Solution:

$$\begin{aligned}\sum_{k=1}^{100} k(k+1) &= \frac{1}{3}\sum_{k=1}^{100}[k(k+1)(k+2)-(k-1)k(k+1)]\\ &= \frac{1}{3}[100\cdot 101\cdot 102 - 0]\\ &= 343400.\end{aligned}$$

4(2). (**Hint:** $a_n - \frac{1}{2}$ is a geometric sequence). $a_n = 3^{n-1}(a_1 - \frac{1}{2}) + \frac{1}{2}$.

Exercise 7.2.6

1(2). Proof. For any $\epsilon > 0$, we choose $\delta = \min(\epsilon, 1)$. Then, for any $0 < |x-4| < \delta$, we know that $x > 0$, and $\sqrt{x}+2 > 2$. Further,

$$|\sqrt{x}-2| = \frac{|x-4|}{\sqrt{x}+2} < \frac{\epsilon}{2} < \epsilon.$$

Thus, by the definition of the limit,

$$\lim_{x\to 4}\sqrt{x} = 2.$$

2(2). (i). $\frac{2}{3}$; (ii) $\frac{2}{3}$.

3(1). It can be proved by mathematical induction.

(2). Proof. Let $a_n = 0$, $b_n = \frac{n}{2^n}$ and $c_n = \frac{1}{n}$. Using the inequality in 3(1), we know that for $n \geq 4$,

$$a_n \leq b_n \leq c_n.$$

Also, it is clear that

$$\lim_{n\to\infty} 0 = 0, \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n} = 0.$$

By the squeeze theorem, we conclude that $\lim_{n\to\infty}\frac{n}{2^n} = 0$.

4(1).Proof. $e > 2$, thus $0 < \frac{n}{e^n} < \frac{n}{2^n}$. Using the squeeze theorem and the result obtained in 3(2), we conclude

$$\lim_{n\to\infty}\frac{n}{e^n} = 0.$$

(2). Proof. Let $m = \ln n$. Then, using the result obtained in 4(1), we know

$$\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{m\to\infty}\frac{m}{e^m} = 0.$$

Exercise 7.3.3

1(2). $1 - 2^{-10} = \frac{1023}{1024}$; (2). 0.

2(2). Divergent; 2(4). Divergent.

-

Exercise 7.4.1

1. $(\frac{k}{2} - \frac{1}{4}) \cdot 3^k + \frac{1}{4}$.

2(2). 2.

Exercise 8.1.1

1(2). $\frac{1}{2}$.

2(2). (i). $v(t) = at$; (ii). $a(t) = a$.

Exercise 8.2.4

1(2). (i). $-2\sin 2x$; (ii). $2x\cos x^2$

(4). (i). $\sec x = \frac{1}{\cos x}$; (ii). $\frac{e^x}{x} - \frac{e^x}{x^2}$.

2(2). (i). Proof: Since, for $x \geq 0$,

$$\frac{d}{dx}(e^x - 1 - x) = e^x - 1 \geq 0,$$

we know function $f(x) = e^x - 1 - x$ is an increasing function for $x \geq 0$. Thus $f(x) \geq f(0) = 0$. It follows that

$$e^x \geq 1 + x.$$

(ii). Proof. Since, for $x \geq 0$,

$$\frac{d}{dx}(x - \sin x) = 1 - \cos x \geq 0,$$

we know function $f(x) = x - \sin x$ is an increasing function for $x \geq 0$. Thus $f(x) \geq f(0) = 0$. It follows that

$$x \geq \sin x.$$

(4). Proof. Since,

$$\frac{d}{dx}(\ln x + \frac{1}{x} - 1) = \frac{1}{x} - \frac{1}{x^2},$$

we know function $f(x) = \ln x + \frac{1}{x} - 1$ is an increasing function for $x \in (0, 1)$, and is a decreasing function for $x \in (1, \infty)$. Thus $0 = f(1) \leq f(x)$ for $x \in (0, \infty)$. It follows that, for $x \in (0, \infty)$,

$$0 \leq \ln x + \frac{1}{x} - 1,$$

that is:

$$\ln x + \frac{1}{x} \geq 1.$$

4. (1). It is slightly easier to show that $y = (x + 1)[\ln(x + 1) - \ln x]$ is decreasing.

(2).

$$(1+\frac{1}{x})^{x+1} \le (1+\frac{1}{5})^6 < 3.$$

(3). It is slightly easier to show that $y = x[\ln(x+1) - \ln x]$ is increasing.

In fact, for $x > 0$,

$$g(x) = \frac{d}{dx}x[\ln(x+1) - \ln x] = \ln(x+1) - \ln x - \frac{1}{x+1},$$

and

$$g'(x) = -\frac{1}{x(x+1)} + \frac{1}{(x+1)^2} < 0$$

. Notice that $\lim_{x\to\infty} g(x) = 0$, so, $g(x) > 0$ for all $x \ge 0$. Thus $y = x[\ln(x+1) - \ln x]$ is increasing.

(4). $\lim_{n\to\infty}(1+\frac{1}{n})^n \ge 2$ since $(1+\frac{1}{n})^n$ is increasing in n.

On the other hand, $\lim_{n\to\infty}(1+\frac{1}{n})^n \le \lim_{n\to\infty}(1+\frac{1}{n})^{n+1} \le (1+\frac{1}{5})^6 < 3$, since $(1+\frac{1}{n})^{n+1}$ is decreasing in n.

Exercise 8.3.1

1(2). (i). $\frac{e^{2x}}{2} + C$; (ii). $-\frac{\cos 2x}{4} + C$.

(4). (i). $\frac{3}{10}(x^2+1)^{\frac{5}{3}} + C$; (ii). $x + 2\ln|x-1| + C$.

2(2). (i). $\ln|\frac{x-1}{x}| + C$; (ii). $\frac{1}{5}\ln|\frac{x-4}{x+1}| + C$.

(4). (i). $-x\cos x + \sin x + C$; (ii). $\frac{e^x \sin x + e^x \cos x}{2} + C$.

Exercise 8.4.3

1(2). (i). 2; (ii). 0.

(4). (i) 0; (ii). 1.

2(2). (i). $v(10) = 50$; (ii). $s(10) = \frac{1000}{6}$.

(4). $v(r) = \frac{4}{3}\pi r^2$.

Exercise 8.5.1 1(2). $2\ln 2 - 2$.

(3). $y = \ln x$ is a convex function, so

$$\begin{aligned}\ln(\frac{x_1+x_2+3}{3}) &= \ln(\frac{(x_1+1)+(x_2+1)+1}{3}) \\ &\ge \frac{\ln(x_1+1)+\ln(x_2+1)+\ln 1}{3} \\ &= \frac{\ln(x_1+1)+\ln(x_2+1)}{3}.\end{aligned}$$

2(2). First, we have

$$a - \frac{b}{x} + \frac{c}{x^2} - \frac{d}{x^3} + \frac{e}{x^4} - \frac{f}{x^5} + \frac{g}{x^6} = 1 + \frac{2}{x} + \frac{3}{x^2} + \frac{4}{x^3} + \frac{5}{x^4} + \frac{6}{x^5} + \frac{7}{x^6}.$$

Sending $x \longrightarrow \infty$, we obtain $a = 1$.

Subtracting 1 in both sides of the above equation and then multiplying x, we have

$$-b + \frac{c}{x} - \frac{d}{x^2} + \frac{e}{x^3} - \frac{f}{x^4} + \frac{g}{x^5} = 2 + \frac{3}{x} + \frac{4}{x^2} + \frac{5}{x^3} + \frac{6}{x^4} + \frac{7}{x^5}.$$

Sending $x \longrightarrow \infty$, we obtain $b = -2$.

Iterating the above processing, we obtain $c = 3,\ d = -4,\ e = 5,\ f = -6,\ g = 7$.

2(4). Proof. If $f(x) = constant$, we know from the definition of derivative that $f'(x) = 0$.

If $f'(x) = 0$, we can prove that $f = constant$ by a contradiction argument. If not, there are two different points $x_1, x_2 \in I$, such that $f(x_1) \neq f(x_2)$. Then by the Mean Value Theorem (Theorem 8.6) there is a point $c \in (x_1, x_2)$, such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

Contradiction.

4(2). (i).

$$\begin{aligned}
1 \times 2 + 2 \times 3 + &\cdots + 99 \times 100 \\
&= \frac{1}{3} \times \{(1 \times 2 \times 3 - 0 \times 1 \times 2) + \cdots + (99 \times 100 \times 101 - 98 \times 99 \times 100)\} \\
&= \frac{1}{3} \times (99 \times 100 \times 101) \\
&= 333300.
\end{aligned}$$

(ii). First, we have

$$\begin{aligned}
\sum_{k=1}^{n} k(k+1) &= \frac{1}{3} \sum_{k=1}^{n} \{k(k+1)(k+2) - (k-1)k(k+1)\} \\
&= \frac{n(n+1)(n+2)}{3}.
\end{aligned}$$

And

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

So,

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} k(k+1) - \sum_{k=1}^{n} k \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$