Zhu on elementary math -16

Solving quadratic equations -2 (in real number set)

EOM

3-27-2021 (updated 10-1-2023)

This article is suitable for students in seventh grade and above and their parents, primary school teachers, middle school teachers, young adults, and any mathematics educator who wants to learn elementary mathematics again.

We mentioned in ``Solving quadratic equations -1'': If a quadratic equation of one variable can be factorized, for example, x^2+x-6=0 is equivalent to (x-3)(x+2)=0, we can then solve the equation based on the properties of equality. This also shows that learning (or being familiar with) factorization is crucial to learning how to solve quadratic equations of one variable.

Unfortunately (in fact, the emergence of a difficulty also heralds the emergence of a breakthrough), some quadratic polynomials cannot be factorized in the field of rational numbers. For example: there is no way to factorize x^2+8x+14 in the field of rational numbers. Therefore, using the factorization method to solve x^2+8x+14=0 in rational number set will hit a wall!

Is it because we did not work hard enough (have you ever heard of "A Foolish Old Man Moving Mountains")? Or we do need new tools?

This is not a matter of hard work or lack of effort. To move mountains, The Foolish Old Man also needs a hard mountain ax and even explosives. Hitting the wall means that we are about to make a leap: we can't find a solution in rational number set, so we have to expand the number field. (Remember how we expanded the number field from natural numbers to rational numbers step by step through algebraic operations?)

First let's look at a slightly simpler equation. Let us solve the equation: x^2=2.

It is equivalent to $x^2=2$. We must ask: Is there number x whose square is 2? There are no such positive integers: 1 is smaller and 2 is larger. What about the fractions: 1.41 is smaller, but 1.42 is larger; 1.414 is still smaller, but 1,415 is still larger!

In fact, we have encountered an invisible mountain: Is there a positive number whose square is 2? This is quite a fundamental mathematical question, and it requires certain mathematical knowledge to answer it. For the sake of simplicity, and for the completeness of the real number field, we admit that there is a positive number x whose square is 2. Let's write this number as $\sqrt{2}$ and call it the principal square root of 2. It can be checked (using operation rules) that the square of $-\sqrt{2}$ is also 2. And yes, we can show that: the principal square root of 2 is not a fraction, that is, it is not a rational number. We call it an irrational number!

With the introduction of square roots, we can solve equations in the real number field (rational numbers + irrational numbers). Let's see how to solve the equation mentioned above.

Example 1. Solve the equation:

$$x^2 + 8x + 14 = 0.$$

Solution: First, let's learn the technique of completing the square --- this is a commonly used technique in mathematics. The purpose is to write the original equation to:

Unknown term ^{2}= constant.

Observe $x^2 + 8x + 14 = 0$ *is equivalent to* $x^2 + 8x + 16 = 2$， *that is,* $(x + 4)^2 = 2$. *So*

$$x + 4 = \sqrt{2}, \text{ or } x + 4 = -\sqrt{2}.$$

From this, we thus obtain

$$x = -4 + \sqrt{2},\ \ or\ \ x = -4 - \sqrt{2}.$$

The above solution is very general: equations that were originally solved using factorization can also be solved using the completing the square method. In fact, for general quadratic equations of one variable

$$x^2 + px + q = 0,$$

using the method of completing the square, we know it is equivalent to

$$(x + \frac{p}{2})^2 = -q + \frac{p^2}{4}.$$

From this we get the famous quadratic formula for the quadratic equation of one variable:

$$x = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

provided that $p^2 - 4q \geq 0$.

We also noticed that Professor Po-Shen Loh introduced a seemingly "concise" calculation technique: First, write $x^2 + px + q$ as

$$x^2 + px + q = (x + \frac{p}{2} + u)(x + \frac{p}{2} - u),$$

where u is the number we are looking for next. Comparing both sides (i.e., quickly, taking x to be 0), we know $q = \left(\frac{p}{2} + u\right)\left(\frac{p}{2} - u\right)$. Thus

$$u^2 = \frac{p^2}{4} - q.$$

From this we get the quadratic formula again:

$$x = -\frac{p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}.$$

Thus, he claimed to obtain another derivation of the quadratic formula.

Here, we would like to critically point out that the above technique can be used as an exercise in solving equations, but should not be regarded as "another derivation" of the quadratic formula. Because: it implicitly uses a more basic theory : Fundamental Theorem of Algebra---This theory guarantees that any quadratic polynomial can always be factorized into the multiplication of two linear polynomials---We will discuss this again later.

Here is another natural but profound question (did you think about it yourself?): if $p^2 - 4q < 0$， how do we solve the above equation? Yes, we are expanding the number field again! First, we discuss: Is there a number x whose square is -1? We sure can simply the introduce the imaginary number i, whose square is defined as -1.

It seems that the problem is solved. Wait! Here is a more difficult question: Is there a number x whose square is i? If you are told that there is a number of the form a+bi (where a and b are two real numbers), whose square is i, then are you confident in figuring out which number is a and which number is b?