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Preface

The purpose of this note is to encourage every high school student to learn complex numbers. My goal is to explain why complex numbers are essential, why every student should develop a solid mastery of them, and why this knowledge becomes even more valuable in the age of artificial intelligence.

Across the world, students are drifting further and further away from complex numbers. This trend is ironic—not because complex numbers are too difficult, and certainly not because they are useless. In fact, they are neither. Rather, students are kept away from complex numbers for several structural reasons rooted in how mathematics is commonly taught.

First, quadratic equations are often not presented in a coherent and conceptually complete way. In many curricula, complex numbers appear only briefly and superficially, usually as a technical device needed to complete the solution of certain equations. They are rarely introduced as a natural extension of the number system itself. As a result, students see complex numbers as artificial symbols rather than as meaningful mathematical objects.

Second, trigonometric functions typically enter the curriculum too late. Even when they are introduced, they are often disconnected from geometry on the unit circle, from vectors, and from coordinate geometry. Without these connections, students struggle to visualize the complex plane and to understand what complex numbers actually represent. The geometric meaning of multiplication, rotation, and magnitude is thus lost.

Third, De Moivre's formula often appears mysterious and poorly motivated. Its derivation relies on trigonometric addition identities that are unintuitive for most learners. Although these identities are easy for computers to implement, they are not natural for humans; nevertheless, students are frequently asked to memorize and apply them by rote, with little conceptual grounding or geometric insight.

In this note, I aim to show that none of these obstacles is fundamental. They are artifacts of pedagogy, not intrinsic difficulties of the subject itself. Once these barriers are removed, the world of complex numbers reveals itself to be remarkably natural, elegant, and indispensable. Indeed, this so-called "imaginary" world is often more coherent and more intuitive than the world of real numbers alone.

We begin in Chapter 1 with the familiar topic of quadratic equations. By examining their solutions carefully, we uncover the natural motivation for introducing imaginary numbers—a line of reasoning that can be traced back about two thousand years to Hero of Alexandria. Along the way, we briefly review the historical evolution of methods for solving quadratic equations, emphasizing how algebraic techniques developed in response to genuine mathematical needs.

As is widely discussed in the literature, attempts to find a general formula for solving cubic equations forced mathematicians to confront expressions involving square roots of negative numbers. This historical struggle forms the central theme of Chapter 2. There we see that complex numbers did not arise from abstraction for its own sake, but from necessity—a recurring pattern in the history of mathematics.

In Chapter 3, we formally introduce imaginary and complex numbers, together with their arithmetic operations. Naturally[1], we assume that the familiar operation rules for real numbers—commutativity, associativity, and distributivity—continue to hold for complex numbers. For example,

[1] Historically, this approach dates back to Bombelli's computations in the sixteenth century.

the identity

$$2i + 4i = 6i$$

is not a definition, but a consequence of the distributive rule:

$$\begin{aligned} 2i + 4i &= (2 + 4)i \\ &= 6i. \end{aligned}$$

Once we adopt these basic operational principles, solving equations such as

$$x^2 = -15$$

poses no difficulty. More importantly, the framework of complex numbers allows us to express the complete set of solutions of all quadratic and cubic equations with real coefficients—an achievement that is impossible within the real-number system alone. These ideas form the main content of Chapter 4. At this stage, however, we have to postpone discussing exponential rules due to technical reasons.

Questions such as

$$x^3 = i$$

—or even whether such equations have solutions at all—lead naturally to the polar form of complex numbers and to their geometric interpretation. In Chapter 5, we present the classical De Moivre formula: elegant in its final form, yet historically difficult to justify. We then introduce an alternative derivation based on Ptolemy's Theorem, leading to a refinement that we call the Ptolemy–De Moivre formula. Using these ideas, we establish the Fundamental Theorem of Algebra for cubic equations. At the end of the chapter, we introduce the exponential rule for complex exponents with the aid of Euler's formula (without proof at this stage). By this point, the reader will have a comprehensive understanding of complex numbers and their operations.

Chapter 6 is devoted to one of the most beautiful results in mathematics: Euler's formula. Some of the topics may challenge readers without prior exposure to calculus, as ideas such as limits and derivatives naturally arise. Nevertheless, the discussion is guided by common sense and intuition. We explain why the number e is truly "natural"—why its rate of growth coincides with the value of the function itself—and how Taylor series emerge naturally as a means of approximating functions by polynomials. The chapter concludes with an introduction to Fourier series, the mathematical foundation of the digital world, including artificial intelligence, highlighting the profound and modern significance of Euler's formula.

Finally, the appendix provides a concise summary of the first principles governing addition, multiplication, and exponentiation of complex numbers, serving as a convenient reference for the reader.