

## Exercises for Section 1.1

1. If  $A$  and  $B$  are sets, show that  $A \subseteq B$  if and only if  $A \cap B = A$ .
2. Prove the second De Morgan Law [Theorem 1.1.4(b)].
3. Prove the Distributive Laws:
  - (a)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ ,
  - (b)  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .
4. The **symmetric difference** of two sets  $A$  and  $B$  is the set  $D$  of all elements that belong to either  $A$  or  $B$  but not both. Represent  $D$  with a diagram.
  - (a) Show that  $D = (A \setminus B) \cup (B \setminus A)$ .
  - (b) Show that  $D$  is also given by  $D = (A \cup B) \setminus (A \cap B)$ .
5. For each  $n \in \mathbb{N}$ , let  $A_n = \{(n+1)k : k \in \mathbb{N}\}$ .
  - (a) What is  $A_1 \cap A_2$ ?
  - (b) Determine the sets  $\bigcup \{A_n : n \in \mathbb{N}\}$  and  $\bigcap \{A_n : n \in \mathbb{N}\}$ .
6. Draw diagrams in the plane of the Cartesian products  $A \times B$  for the given sets  $A$  and  $B$ .
  - (a)  $A = \{x \in \mathbb{R} : 1 \leq x \leq 2 \text{ or } 3 \leq x \leq 4\}$ ,  $B = \{x \in \mathbb{R} : x = 1 \text{ or } x = 2\}$ .
  - (b)  $A = \{1, 2, 3\}$ ,  $B = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$ .
7. Let  $A := B := \{x \in \mathbb{R} : -1 \leq x \leq 1\}$  and consider the subset  $C := \{(x, y) : x^2 + y^2 = 1\}$  of  $A \times B$ . Is this set a function? Explain.
8. Let  $f(x) := 1/x^2$ ,  $x \neq 0$ ,  $x \in \mathbb{R}$ .
  - (a) Determine the direct image  $f(E)$  where  $E := \{x \in \mathbb{R} : 1 \leq x \leq 2\}$ .
  - (b) Determine the inverse image  $f^{-1}(G)$  where  $G := \{x \in \mathbb{R} : 1 \leq x \leq 4\}$ .
9. Let  $g(x) := x^2$  and  $f(x) := x + 2$  for  $x \in \mathbb{R}$ , and let  $h$  be the composite function  $h := g \circ f$ .
  - (a) Find the direct image  $h(E)$  of  $E := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ .
  - (b) Find the inverse image  $h^{-1}(G)$  of  $G := \{x \in \mathbb{R} : 0 \leq x \leq 4\}$ .
10. Let  $f(x) := x^2$  for  $x \in \mathbb{R}$ , and let  $E := \{x \in \mathbb{R} : -1 \leq x \leq 0\}$  and  $F := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ . Show that  $E \cap F = \{0\}$  and  $f(E \cap F) = \{0\}$ , while  $f(E) = f(F) = \{y \in \mathbb{R} : 0 \leq y \leq 1\}$ . Hence  $f(E \cap F)$  is a proper subset of  $f(E) \cap f(F)$ . What happens if 0 is deleted from the sets  $E$  and  $F$ ?
11. Let  $f$  and  $E, F$  be as in Exercise 10. Find the sets  $E \setminus F$  and  $f(E) \setminus f(F)$  and show that it is *not* true that  $f(E \setminus F) \subseteq f(E) \setminus f(F)$ .
12. Show that if  $f : A \rightarrow B$  and  $E, F$  are subsets of  $A$ , then  $f(E \cup F) = f(E) \cup f(F)$  and  $f(E \cap F) \subseteq f(E) \cap f(F)$ .
13. Show that if  $f : A \rightarrow B$  and  $G, H$  are subsets of  $B$ , then  $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$  and  $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$ .
14. Show that the function  $f$  defined by  $f(x) := x/\sqrt{x^2 + 1}$ ,  $x \in \mathbb{R}$ , is a bijection of  $\mathbb{R}$  onto  $\{y : -1 < y < 1\}$ .
15. For  $a, b \in \mathbb{R}$  with  $a < b$ , find an explicit bijection of  $A := \{x : a < x < b\}$  onto  $B := \{y : 0 < y < 1\}$ .
16. Give an example of two functions  $f, g$  on  $\mathbb{R}$  to  $\mathbb{R}$  such that  $f \neq g$ , but such that  $f \circ g = g \circ f$ .
17.
  - (a) Show that if  $f : A \rightarrow B$  is injective and  $E \subseteq A$ , then  $f^{-1}(f(E)) = E$ . Give an example to show that equality need not hold if  $f$  is not injective.
  - (b) Show that if  $f : A \rightarrow B$  is surjective and  $H \subseteq B$ , then  $f(f^{-1}(H)) = H$ . Give an example to show that equality need not hold if  $f$  is not surjective.
18.
  - (a) Suppose that  $f$  is an injection. Show that  $f^{-1} \circ f(x) = x$  for all  $x \in D(f)$  and that  $f \circ f^{-1}(y) = y$  for all  $y \in R(f)$ .
  - (b) If  $f$  is a bijection of  $A$  onto  $B$ , show that  $f^{-1}$  is a bijection of  $B$  onto  $A$ .

19. Prove that if  $f : A \rightarrow B$  is bijective and  $g : B \rightarrow C$  is bijective, then the composite  $g \circ f$  is a bijective map of  $A$  onto  $C$ .
20. Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be functions.  
 (a) Show that if  $g \circ f$  is injective, then  $f$  is injective.  
 (b) Show that if  $g \circ f$  is surjective, then  $g$  is surjective.
21. Prove Theorem 1.1.14.
22. Let  $f, g$  be functions such that  $(g \circ f)(x) = x$  for all  $x \in D(f)$  and  $(f \circ g)(y) = y$  for all  $y \in D(g)$ . Prove that  $g = f^{-1}$ .

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## Section 1.2 Mathematical Induction

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Mathematical Induction is a powerful method of proof that is frequently used to establish the validity of statements that are given in terms of the natural numbers. Although its utility is restricted to this rather special context, Mathematical Induction is an indispensable tool in all branches of mathematics. Since many induction proofs follow the same formal lines of argument, we will often state only that a result follows from Mathematical Induction and leave it to the reader to provide the necessary details. In this section, we will state the principle and give several examples to illustrate how inductive proofs proceed.

We shall assume familiarity with the set of natural numbers:

$$\mathbb{N} := \{1, 2, 3, \dots\},$$

with the usual arithmetic operations of addition and multiplication, and with the meaning of a natural number being less than another one. We will also assume the following fundamental property of  $\mathbb{N}$ .

### 1.2.1 Well-Ordering Property of $\mathbb{N}$ Every nonempty subset of $\mathbb{N}$ has a least element.

A more detailed statement of this property is as follows: If  $S$  is a subset of  $\mathbb{N}$  and if  $S \neq \emptyset$ , then there exists  $m \in S$  such that  $m \leq k$  for all  $k \in S$ .

On the basis of the Well-Ordering Property, we shall derive a version of the Principle of Mathematical Induction that is expressed in terms of subsets of  $\mathbb{N}$ .

### 1.2.2 Principle of Mathematical Induction Let $S$ be a subset of $\mathbb{N}$ that possesses the two properties:

- (1) The number  $1 \in S$ .
- (2) For every  $k \in \mathbb{N}$ , if  $k \in S$ , then  $k + 1 \in S$ .

Then we have  $S = \mathbb{N}$ .

**Proof.** Suppose to the contrary that  $S \neq \mathbb{N}$ . Then the set  $\mathbb{N} \setminus S$  is not empty, so by the Well-Ordering Principle it has a least element  $m$ . Since  $1 \in S$  by hypothesis (1), we know that  $m > 1$ . But this implies that  $m - 1$  is also a natural number. Since  $m - 1 < m$  and since  $m$  is the least element in  $\mathbb{N}$  such that  $m \notin S$ , we conclude that  $m - 1 \in S$ .

We now apply hypothesis (2) to the element  $k := m - 1$  in  $S$ , to infer that  $k + 1 = (m - 1) + 1 = m$  belongs to  $S$ . But this statement contradicts the fact that  $m \notin S$ . Since  $m$  was obtained from the assumption that  $\mathbb{N} \setminus S$  is not empty, we have obtained a contradiction. Therefore we must have  $S = \mathbb{N}$ . Q.E.D.

This is the formula for the sum of the terms in a “geometric progression”. It can be established using Mathematical Induction as follows. First, if  $n = 1$ , then  $1 + r = (1 - r^2)/(1 - r)$ . If we assume the truth of the formula for  $n = k$  and add the term  $r^{k+1}$  to both sides, we get (after a little algebra)

$$1 + r + r^k + \dots + r^{k+1} = \frac{1 - r^{k+1}}{1 - r} + r^{k+1} = \frac{1 - r^{k+2}}{1 - r},$$

which is the formula for  $n = k + 1$ . Therefore, Mathematical Induction implies the validity of the formula for all  $n \in \mathbb{N}$ .

[This result can also be proved without using Mathematical Induction. If we let  $s_n := 1 + r + r^2 + \dots + r^n$ , then  $rs_n = r + r^2 + \dots + r^{n+1}$ , so that

$$(1 - r)s_n = s_n - rs_n = 1 - r^{n+1}.$$

If we divide by  $1 - r$ , we obtain the stated formula.]

(g) Careless use of the Principle of Mathematical Induction can lead to obviously absurd conclusions. The reader is invited to find the error in the “proof” of the following assertion.

**Claim:** If  $n \in \mathbb{N}$  and if the maximum of the natural numbers  $p$  and  $q$  is  $n$ , then  $p = q$ .

“Proof.” Let  $S$  be the subset of  $\mathbb{N}$  for which the claim is true. Evidently,  $1 \in S$  since if  $p, q \in \mathbb{N}$  and their maximum is 1, then both equal 1 and so  $p = q$ . Now assume that  $k \in S$  and that the maximum of  $p$  and  $q$  is  $k + 1$ . Then the maximum of  $p - 1$  and  $q - 1$  is  $k$ . But since  $k \in S$ , then  $p - 1 = q - 1$  and therefore  $p = q$ . Thus,  $k + 1 \in S$ , and we conclude that the assertion is true for all  $n \in \mathbb{N}$ .

(h) There are statements that are true for many natural numbers but that are not true for all of them.

For example, the formula  $p(n) := n^2 - n + 41$  gives a prime number for  $n = 1, 2, \dots, 40$ . However,  $p(41)$  is obviously divisible by 41, so it is not a prime number.  $\square$

Another version of the Principle of Mathematical Induction is sometimes quite useful. It is called the “Principle of Strong Induction”, even though it is in fact equivalent to 1.2.2.

**1.2.5 Principle of Strong Induction** Let  $S$  be a subset of  $\mathbb{N}$  such that

(1'')  $1 \in S$ .

(2'') For every  $k \in \mathbb{N}$ , if  $\{1, 2, \dots, k\} \subseteq S$ , then  $k + 1 \in S$ .

Then  $S = \mathbb{N}$ .

We will leave it to the reader to establish the equivalence of 1.2.2 and 1.2.5.

**Exercises for Section 1.2**

1. Prove that  $1/1 \cdot 2 + 1/2 \cdot 3 + \dots + 1/n(n + 1) = n/(n + 1)$  for all  $n \in \mathbb{N}$ .
2. Prove that  $1^3 + 2^3 + \dots + n^3 = [\frac{1}{2}n(n + 1)]^2$  for all  $n \in \mathbb{N}$ .
3. Prove that  $3 + 11 + \dots + (8n - 5) = 4n^2 - n$  for all  $n \in \mathbb{N}$ .
4. Prove that  $1^2 + 3^2 + \dots + (2n - 1)^2 = (4n^3 - n)/3$  for all  $n \in \mathbb{N}$ .
5. Prove that  $1^2 - 2^2 + 3^2 + \dots + (-1)^{n+1}n^2 = (-1)^{n+1}n(n + 1)/2$  for all  $n \in \mathbb{N}$ .

6. Prove that  $n^3 + 5n$  is divisible by 6 for all  $n \in \mathbb{N}$ .
7. Prove that  $5^{2n} - 1$  is divisible by 8 for all  $n \in \mathbb{N}$ .
8. Prove that  $5^n - 4n - 1$  is divisible by 16 for all  $n \in \mathbb{N}$ .
9. Prove that  $n^3 + (n+1)^3 + (n+2)^3$  is divisible by 9 for all  $n \in \mathbb{N}$ .
10. Conjecture a formula for the sum  $1/1 \cdot 3 + 1/3 \cdot 5 + \cdots + 1/(2n-1)(2n+1)$ , and prove your conjecture by using Mathematical Induction.
11. Conjecture a formula for the sum of the first  $n$  odd natural numbers  $1 + 3 + \cdots + (2n-1)$ , and prove your formula by using Mathematical Induction.
12. Prove the Principle of Mathematical Induction 1.2.3 (second version).
13. Prove that  $n < 2^n$  for all  $n \in \mathbb{N}$ .
14. Prove that  $2^n < n!$  for all  $n \geq 4, n \in \mathbb{N}$ .
15. Prove that  $2n - 3 \leq 2^{n-2}$  for all  $n \geq 5, n \in \mathbb{N}$ .
16. Find all natural numbers  $n$  such that  $n^2 < 2^n$ . Prove your assertion.
17. Find the largest natural number  $m$  such that  $n^3 - n$  is divisible by  $m$  for all  $n \in \mathbb{N}$ . Prove your assertion.
18. Prove that  $1/\sqrt{1} + 1/\sqrt{2} + \cdots + 1/\sqrt{n} > \sqrt{n}$  for all  $n \in \mathbb{N}$ .
19. Let  $S$  be a subset of  $\mathbb{N}$  such that (a)  $2^k \in S$  for all  $k \in \mathbb{N}$ , and (b) if  $k \in S$  and  $k \geq 2$ , then  $k-1 \in S$ . Prove that  $S = \mathbb{N}$ .
20. Let the numbers  $x_n$  be defined as follows:  $x_1 := 1$ ,  $x_2 := 2$ , and  $x_{n+2} := \frac{1}{2}(x_{n+1} + x_n)$  for all  $n \in \mathbb{N}$ . Use the Principle of Strong Induction (1.2.5) to show that  $1 \leq x_n \leq 2$  for all  $n \in \mathbb{N}$ .

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### Section 1.3 Finite and Infinite Sets

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When we count the elements in a set, we say “one, two, three, . . .”, stopping when we have exhausted the set. From a mathematical perspective, what we are doing is defining a bijective mapping between the set and a portion of the set of natural numbers. If the set is such that the counting does not terminate, such as the set of natural numbers itself, then we describe the set as being infinite.

The notions of “finite” and “infinite” are extremely primitive, and it is very likely that the reader has never examined these notions very carefully. In this section we will define these terms precisely and establish a few basic results and state some other important results that seem obvious but whose proofs are a bit tricky. These proofs can be found in Appendix B and can be read later.

- 1.3.1 Definition** (a) The empty set  $\emptyset$  is said to have 0 elements.
- (b) If  $n \in \mathbb{N}$ , a set  $S$  is said to have  $n$  elements if there exists a bijection from the set  $\mathbb{N}_n := \{1, 2, \dots, n\}$  onto  $S$ .
- (c) A set  $S$  is said to be **finite** if it is either empty or it has  $n$  elements for some  $n \in \mathbb{N}$ .
- (d) A set  $S$  is said to be **infinite** if it is not finite.

Since the inverse of a bijection is a bijection, it is easy to see that a set  $S$  has  $n$  elements if and only if there is a bijection from  $S$  onto the set  $\{1, 2, \dots, n\}$ . Also, since the composition of two bijections is a bijection, we see that a set  $S_1$  has  $n$  elements if and only