Some general forms of sharp Sobolev inequalities

Meijun Zhu
Department of Mathematics
The University of British Columbia
Vancouver, B.C. Canada V6T 1Z2
mzhu@math.ubc.ca

Abstract

In this paper, we establish some general forms of sharp Sobolev inequalities on the upper half space or any compact Riemannian manifold with smooth boundary. These forms extend some previous results due to Escobar [11], Li and Zhu [18].

0 Introduction

In the past several decades, the study of sharp Sobolev inequalities has attracted the attention of many mathematicians. Not only do those sharp type Sobolev inequalities play essential roles in the study of some problems arising from geometry and physics, but also those inequalities themselves indicate some rich and significant phenomena (for example, under which circumstance the extremal functions for those inequalities exist).

The sharp Sobolev inequalities for functions vanishing on the boundary are well understood for manifolds with dimension greater than or equal to 3, see Aubin [3], Talenti [21], Brezis-Nirenberg [6], Lieb [19], Hebey and Vaugon [14]-[15], Hebey [13], and the references therein. However, it seems that those sharp Sobolev inequalities for functions which do not vanish on the boundary still need to be further studied, even though there are already some interesting results, see Lions [20], Brezis-Lieb [5], Escobar [11]-[12], Beckner [4], Adimurthi and Yadava [2], Carlen and Loss [8]-[9], Li and Zhu [17]-[18], and the references therein. One of the most interesting problems is to study the relations between the L^2 norm of the gradient, the boundary L^q

norm and the interior L^p norm. Here and throughout this paper we always set $q = 2(n-1)/(n-2)$, $p = 2n/(n-2)$, where $n \geq 3$ is the dimension of the manifold.

In this paper, we continue our previous work [17], [18] and give some sharp relations between the three terms we just mentioned. It turns out that some previous results are special cases, see Remark 0.2-0.3 and 0.5 below.

Denote $\mathbb{R}^n_+ = \{(x', x_n) \ : \ x' = (x_1, x_2, ..., x_{n-1}), \ x_n > 0\}$ as the upper half space, and $D^{1,2}(\mathbb{R}^n_+) = \{u \ : \ \int_{\mathbb{R}^n_+} |\nabla u|^2 < \infty, \ \int_{\mathbb{R}^n_+} |u|^p < \infty\}$. Let $1/S = (n-2)/2 \cdot (2\pi^{n/2}/(\Gamma(n/2)))^{1/(n-1)}$ be the sharp constant corresponding to the trace inequality, $Z_0 = (n-2)C_n^{1/(n-1)}$, where $C_n = \int_{\mathbb{R}^{n-1}} (\frac{1}{1+|x|^2})^{n-1} dx = \pi^{(n-1)/2}\Gamma((n-1)/2)/\Gamma(n-1)$. One can check that

$$Z_0 = \frac{1}{S} \tag{0.1}$$

For any $\epsilon > 0$ and $d \in \mathbb{R}$, we define

$$u_{\epsilon,d}(x) = (\frac{\epsilon}{\epsilon^2 + |x'|^2 + |x_n - \epsilon d|^2})^{\frac{n-2}{2}}. \tag{0.2}$$

Considering three typical manifolds, we have corresponding theorems as follows.

On the upper half space, we have

Theorem 0.1 *For any $Z \in (-1/S, Z_0)$, let*

$$d_z = \frac{Z}{\sqrt{Z_0^2 - Z^2}}, \tag{0.3}$$

and $S_1(Z)$ be given by

$$\frac{1}{S_1(Z)} = \frac{\int_{\mathbb{R}^n_+} |\nabla u_{\epsilon,d_z}|^2 + Z(\int_{\partial\mathbb{R}^n_+} u^q_{\epsilon,d_z})^{\frac{2}{q}}}{(\int_{\mathbb{R}^n_+} u^p_{\epsilon,d_z})^{2/p}}. \tag{0.4}$$

Then

$$(\int_{\mathbb{R}^n_+} |u|^p)^{\frac{2}{p}} \leq S_1(Z)\Big(\int_{\mathbb{R}^n_+} |\nabla u|^2 + Z(\int_{\partial\mathbb{R}^n_+} |u|^q)^{\frac{2}{q}}\Big), \quad \forall u \in D^{1,2}(\mathbb{R}^n_+), \tag{0.5}$$

and equality holds if and only if $u(x) = Cu_{\epsilon,d_z}$ for some $\epsilon > 0$, $C \in \mathbb{R}$.

For $Z = Z_0$, let $1/S_1 = \pi n(n-2)(\Gamma(n/2)/\Gamma(n))^{2/n}$ be the sharp constant of Sobolev inequality in $\mathbb{R}^n$. Then

$$(\int_{\mathbb{R}^n_+} |u|^p)^{\frac{2}{p}} < S_1\Big(\int_{\mathbb{R}^n_+} |\nabla u|^2 + Z_0(\int_{\partial\mathbb{R}^n_+} |u|^q)^{\frac{2}{q}}\Big), \quad \forall u \in D^{1,2}(\mathbb{R}^n_+). \tag{0.6}$$

Remark 0.1 *One can check that* $S_1(Z) \to \infty$ *as* $Z \to -1/S$. *An easy consequence of Theorem 0.1 is that* $S_1(Z)$ *strictly decreases from* ∞ *to* $1/S_1$ *as* Z *goes from* $-1/S$ *to* $Z_0 = 1/S$. *The strictness of inequality (0.6) is due to the nonexistence of the extremal function.*

Remark 0.2 *For* $Z \le 0$, *this theorem was proved by Escobar in [11]. It seems to us that his method can not be applied to prove (0.5) for* $Z > 0$.

Let $\Omega \subset \mathbb{R}^n$ $(n \ge 3)$ be a bounded domain, we have

Theorem 0.2 *For any* $Z \in (-1/S, Z_0)$, *let* d_z, $S_1(Z)$ *be given as in Theorem 0.1. Then there exists a constant* $C(Z, \Omega)$ *such that*

$$(\int_\Omega |u|^p)^{\frac{2}{p}} \le S_1(Z)\Big(\int_\Omega |\nabla u|^2 + Z(\int_{\partial\Omega} |u|^q)^{\frac{2}{q}}\Big) + C(Z,\Omega)\int_{\partial\Omega} u^2, \quad \forall u \in H^1(\Omega). \tag{0.7}$$

Remark 0.3 *For* $Z = 0$, *(0.7) was proved by Y.Y.Li and M.Zhu in [18] and was partially proved by Adimurthi and S. L. Yadava in [2]. In this case, one can see that* $S_1(0) = 1/(2^{2/n} S_1)$.

Remark 0.4 *It is interesting to give some upper bound estimates about the constant* $C(Z, \Omega)$. *For a general domain (except a ball), it is hard to say whether the extremal function for (0.7) exists or not.*

Let (M, g) be any compact Riemannian manifold with smooth boundary and dimension $n \ge 3$, we have

Theorem 0.3 *For any* $Z \in (-1/S, Z_0)$, *let* d_z, $S_1(Z)$ *be given as in Theorem 0.1. Then there exists a constant* $D(Z, M)$ *such that* $\forall u \in H^1(M)$

$$\begin{aligned}&(\textstyle\int_M |u|^p dv_g)^{\frac{2}{p}}\\ &\le S_1(Z)\Big(\textstyle\int_M |\nabla_g u|^2 dv_g + Z(\int_{\partial M} |u|^q ds_g)^{\frac{2}{q}}\Big) + D(Z,M)\Big(\int_{\partial M} u^2 ds_g + \int_M u^2 dv_g\Big).\end{aligned} \tag{0.8}$$

Remark 0.5 *When* $Z = 0$, *(0.8) was proved by Y.Y.Li and M.Zhu in [18].*

Remark 0.6 *We do not know whether (0.7) and (0.8) still hold for* $Z = Z_0$ *or not. We tend to believe that it is true, and give the following conjecture.*

Conjecture *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. There exists some constant $C(\Omega) > 0$ such that*

$$(\int_\Omega |u|^p)^{\frac{2}{p}} \le S_1\Big(\int_\Omega |\nabla u|^2 + Z_0(\int_{\partial\Omega} |u|^q)^{\frac{2}{q}}\Big) + C(\Omega)\int_{\partial\Omega} u^2, \quad \forall u \in H^1(\Omega). \qquad (0.9)$$

Let (M, g) be a compact Riemannian manifold with smooth boundary and dimension ≥ 3. There exists some constant $D(M)$ such that $\forall u \in H^1(M)$

$$\begin{aligned}&(\textstyle\int_M |u|^p dv_g)^{\frac{2}{p}}\\ &\le S_1\Big(\textstyle\int_M |\nabla_g u|^2 dv_g + Z_0(\int_{\partial M} |u|^q ds_g)^{\frac{2}{q}}\Big) + D(M)\Big(\textstyle\int_{\partial M} u^2 ds_g + \int_M u^2 dv_g\Big).\end{aligned} \qquad (0.10)$$

Since the upper half space is conformally equivalent to a ball, we know from Theorem 0.1 that (0.9) holds for some constant $C(\Omega)$ when Ω is a ball in $\mathbb{R}^n$.

Remark 0.7 *It can be easily seen that if (0.10) held for some large constant $D(M)$, Hebey and Vaugon's inequality (see [14]) would be its corollary.*

Remark 0.8 *In [5], Brezis and Lieb proved that for any bounded domain $\Omega \subset \mathbb{R}^n$, there exists a constant C, such that*

$$||u||_{p,\Omega} \le S_1^{1/2}||\nabla u||_{2,\Omega} + C||u||_{q,\partial\Omega}, \quad \forall u \in H^1(\Omega).$$

They asked whether there exists a constant C_1, such that

$$||u||^2_{p,\Omega} \le S_1||\nabla u||^2_{2,\Omega} + C_1||u||^2_{q,\partial\Omega}, \quad \forall u \in H^1(\Omega). \qquad (0.11)$$

It is easy to see that (0.11) follows directly from (0.9). Hence the answer to their question is affirmative when the domain is a ball.

The proof of Theorem 0.1 heavily depends on the conformal invariant property of the corresponding energy functionals between the upper half space and the unit ball. The key ingredient is to show that the infimum of the corresponding functional is attained under the assumption of small energy (see Proposition 1.1 below for precise statement). We use a new approach which combines some old ideas (blow-up argument) with some new inequalities initiated by the work of Brezis and Lieb [5](see corollary 1.2 below). The proofs of Theorem 0.2 and 0.3 also involve this difficulty; we overcome it by using the same method. Some ingredients in the proofs of Theorem 0.2 and Theorem 0.3 have already appeared in our previous work [17], [18] and [22].

This paper is organized as follows. In Section 1 we give the proof of Theorem 0.1. In Section 2 we prove Theorem 0.2 through an argument by contradiction . In Section 3 we sketch the proof of Theorem 0.3. In Section 4 we give some discussions concerning the conjecture and point out the obstacle of proving this conjecture by the current method. When I was working on this problem, I was informed by Y.Y.Li about the recent results of Carlen and Loss [9]. It turns out that Theorem 0.1 can also be derived from their results. This derivation is included in the appendix. It seems to me that their method can not be applied to general domains.

Throughout this paper, we use $C_0,\ C,\ C_1, C_2, ...$, to represent some various positive constants, $\epsilon,\ \epsilon_0,\ \epsilon_1, ..., \delta_0,\ \delta,\ \delta_1, ...$, to represent some various small positive constants. Without specific mention, we always pass to a limit up to some subsequence of ϵ or α.

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1 The upper half space and unit ball

For any $Z \in (-1/S, Z_0]$, we define

$$I_Z(u) = \frac{\int_{\mathbb{R}^n_+} |\nabla u|^2 + Z(\int_{\partial\mathbb{R}^n_+} |u|^q)^{2/q}}{(\int_{\mathbb{R}^n_+} |u|^p)^{2/p}}, \quad \forall u \in D^{1,2}(\mathbb{R}^n_+) \setminus \{0\}$$

and

$$II_Z(u) = \frac{\int_{B_1} |\nabla u|^2 + Z(\int_{\partial B_1} |u|^q)^{2/q} + \frac{n-2}{2}\int_{\partial B_1} u^2}{(\int_{B_1} |u|^p)^{2/p}}, \quad \forall u \in H^1(B_1) \setminus \{0\}.$$

Due to the conformal invariance, we know

$$\inf_{u\in D^{1,2}(\mathbb{R}^n_+)\setminus\{0\}} I_Z(u) = \inf_{u\in H^1(B_1)\setminus\{0\}} II_Z(u) := \xi_Z. \tag{1.1}$$

The key step in our proof of Theorem 0.1 is to establish the following proposition.

Proposition 1.1 *If* $0 < \xi_Z < 1/S_1$, *then* $\inf I_Z(u)$ *and* $\inf II_Z(u)$ *are achieved.*

Proof. We prove this proposition by contradiction.

Suppose these infimums are not attained. For any $0 < \epsilon < 1$ and $u \neq 0$, we define

$$II_\epsilon(u) = \frac{\int_{B_1} |\nabla u|^2 + Z(\int_{\partial B_1} |u|^{q_\epsilon})^{2/q_\epsilon} + \frac{n-2}{2}\int_{\partial B_1} u^2}{(\int_{B_1} |u|^{p_\epsilon})^{2/p_\epsilon}}$$

where and throughout this paper, we set $p_\epsilon = p - \epsilon$, $q_\epsilon = q - \epsilon/2$. Denote

$$\xi_\epsilon := \inf_{u \in H^1(B_1)\setminus\{0\}} II_\epsilon(u).$$

In order to derive a contradiction, we need several lemmas.

Lemma 1.1 *As ϵ small enough,*

$$0 \leq \xi_\epsilon \leq \xi_Z.$$

Proof. $\forall \delta > 0$, there exists a $\bar{u} \in C^\infty(B_1)$, such that

$$\frac{\int_{B_1} |\nabla \bar{u}|^2 + Z(\int_{\partial B_1} |\bar{u}|^q)^{2/q} + \frac{n-2}{2}\int_{\partial B_1} \bar{u}^2}{(\int_{B_1} |\bar{u}|^p)^{2/p}} \leq \xi_Z + \delta.$$

Also as $\epsilon \to 0$,

$$\begin{aligned} &\frac{\int_{B_1} |\nabla \bar{u}|^2 + Z(\int_{\partial B_1} |\bar{u}|^q)^{2/q} + \frac{n-2}{2}\int_{\partial B_1} \bar{u}^2}{(\int_{B_1} |\bar{u}|^p)^{2/p}} \\ \geq\; &\frac{\int_{B_1} |\nabla \bar{u}|^2 + Z(\int_{\partial B_1} |\bar{u}|^{q_\epsilon})^{2/q_\epsilon} + \frac{n-2}{2}\int_{\partial B_1} \bar{u}^2}{(\int_{B_1} |\bar{u}|^{p_\epsilon})^{2/p_\epsilon}} - \delta \\ \geq\; &\xi_\epsilon - \delta. \end{aligned}$$

Combining above two inequalities together we know $\xi_\epsilon \leq \xi_Z$, as $\epsilon \to 0$. If $Z \geq 0$, it is obvious that $\xi_\epsilon \geq 0$. If $0 > Z > -1/S$, by sharp trace inequality, we know that for any fixed $0 < \delta_0 < Z/2 + 1/(2S)$,

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 + (Z - \delta_0)(\int_{\partial \mathbb{R}^n_+} |u|^q)^{2/q} \geq 0, \quad \forall\; u \in D^{1,2}(\mathbb{R}^n_+).$$

It follows from the conformal invariance of the energy that

$$\int_{B_1} |\nabla u|^2 + (Z - \delta_0)(\int_{\partial B_1} |u|^q)^{2/q} + \frac{n-2}{2}\int_{\partial B_1} u^2 \geq 0, \; \forall u \in H^1(\Omega).$$

Notice $Z - \delta_0 < 0$. We know from the above and Hölder inequality that as $\epsilon \to 0$, $\xi_\epsilon \geq 0$.

It follows from Lemma 1.1 and the standard variational method that as ϵ sufficiently small, there exists $u_\epsilon \geq 0$ with $||u_\epsilon||_{p_\epsilon, B_1} = 1$ such that

$$II_\epsilon(u_\epsilon) = \inf_{u \in H^1(B_1)\backslash\{0\}} II_\epsilon(u) = \xi_\epsilon.$$

Next lemma is a slight extension of an inequality due to Brezis and Lieb [5].

Lemma 1.2 *Let Ω be a bounded domain in $\mathbb{R}^n$, $|\Omega| = 1$, $1 < r \leq p$, $s = \frac{(n-1)r}{n}$, then there exists a constant $C(\Omega, r)$, such that*

$$(\int_\Omega |f|^r)^{1/r} \leq S_1^{1/2} (\int_\Omega |\nabla f|^2)^{1/2} + C(\Omega, r)(\int_{\partial\Omega} |f|^s)^{1/s}, \ \forall \ f \in H^1(\Omega). \tag{1.2}$$

Further, as r close to $2n/(n-2)$, we can choose $C(\Omega, r)$ independent of r.

Proof. We only need to show that (1.2) holds for any smooth function f. Let h be the solution of the following equation:

$$\begin{cases} -\Delta h = 0 & \text{in } \ \Omega \\ h = f & \text{on } \ \partial\Omega \end{cases}$$

and $u = f - h$. Then $u = 0$ on $\partial\Omega$. Therefore

$$||u||_{r,\Omega} \leq ||u||_{p,\Omega} \leq S_1^{1/2} ||\nabla u||_{2,\Omega}. \tag{1.3}$$

One can easily check that

$$||\nabla u||_{2,\Omega}^2 = ||\nabla f||_{2,\Omega}^2 - ||\nabla h||_{2,\Omega}^2. \tag{1.4}$$

Also, by Minkowski inequality

$$||u||_{r,\Omega} \geq ||f||_{r,\Omega} - ||h||_{r,\Omega}. \tag{1.5}$$

Combining (1.3) and (1.4) with (1.5), we have

$$||f||_{r,\Omega} - ||h||_{r,\Omega} \leq S_1^{1/2} (||\nabla f||_{2,\Omega}^2 - ||\nabla h||_{2,\Omega}^2)^{1/2} \leq S_1^{1/2} ||\nabla f||_{2,\Omega}. \tag{1.6}$$

We claim:

$$||h||_{r,\Omega} \leq C(\Omega, r) ||f||_{s,\partial\Omega}$$

with $C(\Omega, r)$ independent of r as r close to $2n/(n-2)$.

Lemma 1.2 directly follows from the claim and (1.6). Therefore, we are left to prove the claim.

Let ϕ be the solution of the following equation

$$\begin{cases} \Delta\phi = Y & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

for some $Y \in L^t(\Omega)$. Then

$$\int_\Omega hY = \int_{\partial\Omega} h\frac{\partial\phi}{\partial\nu}. \tag{1.7}$$

By elliptic estimates, we have

$$||\partial_{ij}\phi||_{t,\Omega} \leq C_t||Y||_{t,\Omega}, \quad ||\nabla\phi||_{W^{1,t}(\Omega)} \leq C_t||Y||_{t,\Omega},$$

thus

$$||\frac{\partial\phi}{\partial\nu}||_{\beta,\partial\Omega} \leq C_t||Y||_{t,\Omega} \tag{1.8}$$

where $1/\beta = n/((n-1)t) - 1/(n-1)$. Also we can choose a uniform C_t if $2n/(n+2) \leq t \leq 2$. It follows from (1.7) and (1.8) that

$$\int_\Omega hY \leq ||h||_{\beta',\partial\Omega} \cdot ||\frac{\partial\phi}{\partial\nu}||_{\beta,\partial\Omega} \leq C_t||f||_{\beta',\partial\Omega} \cdot ||Y||_{t,\Omega} \tag{1.9}$$

where $1/\beta + 1/\beta' = 1$.

Let $1/r = 1 - 1/t$, then $1/\beta' = n/((n-1)r)$, that is $\beta' = s$. Therefore, by (1.9)

$$\int_\Omega hY \leq C_t||f||_{s,\partial\Omega} \cdot ||Y||_{t,\Omega}.$$

Claim follows from the above directly.

A quick consequence of the above lemma is the following.

Corollary 1.1 *Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $|\Omega| = 1$. There exists a constant $C(\Omega)$, such that as $\epsilon < 1/100$,*

$$||f||_{p_\epsilon,\Omega} \leq S_1^{1/2}||\nabla f||_{2.\Omega} + C(\Omega)||f||_{q_\epsilon,\partial\Omega}, \quad \forall f \in H^1(\Omega). \tag{1.10}$$

Proof. Let $r = p_\epsilon$ in Lemma 1.2, then

$$s = \frac{n-1}{n} \cdot p_\epsilon = q - \frac{n-1}{n} \cdot \epsilon < q_\epsilon.$$

Corollary1.1 follows directly from Hölder inequality.

Later on, we will use this corollary in the following setting. We state it as another corollary.

Corollary 1.2 *Let $\Omega \in \mathbb{R}^n$ be a bounded domain. $\forall \delta > 0$, there exists a constant $C(\Omega, \delta)$, such that as ϵ small enough,*

$$||f||^2_{p_\epsilon,\Omega} \le (S_1 + \delta)||\nabla f||^2_{2,\Omega} + C(\Omega,\delta)||f||^2_{q_\epsilon,\partial\Omega}, \quad \forall f \in H^1(\Omega). \tag{1.11}$$

Proof. Without loss of generality, we assume $\{0\} \in \Omega$ and $|\Omega| = \lambda^n$. Set $\Omega_0 = \Omega/\lambda$ and $f_0(x) = \lambda^{2/(p_\epsilon - 2)} f(\lambda x)$ for $x \in \Omega_0$. Then due to Corollary 1.1, we have, for $\epsilon < 1/100$,

$$||f_0||_{p_\epsilon,\Omega_0} \le S_1^{1/2}||\nabla f_0||_{2.\Omega_0} + C||f_0||_{q_\epsilon,\partial\Omega_0}.$$

By rescaling, we have

$$||f_0||_{p_\epsilon,\Omega_0} = \lambda^{o(1)}||f||_{p_\epsilon,\Omega}, \quad ||\nabla f_0||_{2.\Omega_0} = \lambda^{o(1)}||\nabla f||_{2.\Omega}, \quad ||f_0||_{q_\epsilon,\partial\Omega_0} = \lambda^{o(1)}||f||_{q_\epsilon,\partial\Omega}$$

where $o(1) \to 0$ as $\epsilon \to 0$. Therefore, for any $\delta > 0$, as $\epsilon < \epsilon_1 < 1/100$ for some small ϵ_1, we have

$$||f||_{p_\epsilon,\Omega} \le (S_1^{1/2} + \frac{\delta}{2})||\nabla f||_{2.\Omega} + C_1||f||_{q_\epsilon,\partial\Omega}.$$

Squaring both sides of the above and using Cauchy-Schwartz inequality we have our corollary.

We now continue the proof of Proposition 1.1. Since $II_\epsilon(u_\epsilon) = \xi_\epsilon$ and $||u_\epsilon||_{p_\epsilon,B_1} = 1$, we know that u_ϵ satisfies

$$\begin{cases} -\Delta u_\epsilon &= \xi_\epsilon u_\epsilon^{p_\epsilon - 1} & \text{in } \ B_1 \\ \frac{\partial u_\epsilon}{\partial \nu} &= -Z(\int_{\partial B_1} u_\epsilon^{q_\epsilon})^{2/q_\epsilon - 1} u_\epsilon^{q_\epsilon - 1} - \frac{n-2}{2} u_\epsilon & \text{on } \ \partial B_1. \end{cases} \tag{1.12}$$

Due to Cherrier [10], we know that u_ϵ is smooth up to the boundary. Hence we can assume $u_\epsilon(x_\epsilon) = ||u||_{L^\infty(B_1)}$ for some $x_\epsilon \in \overline{B}_1$. Since $\inf II_Z$ is not attained, we know that $u_\epsilon(x_\epsilon) \to \infty$ as $\epsilon \to 0$.

Define

$$\begin{cases} \mu_\epsilon &= (u_\epsilon(x_\epsilon))^{-\frac{p_\epsilon - 2}{2}}, \quad \Omega_\epsilon = \frac{B_1 - x_\epsilon}{\mu_\epsilon}, \\ v_\epsilon(x) &= \mu_\epsilon^{\frac{2}{p_\epsilon - 2}} u_\epsilon(\mu_\epsilon x + x_\epsilon) \ \text{ for } \ x \in \Omega_\epsilon. \end{cases} \tag{1.13}$$

Then v_ϵ satisfies

$$\begin{cases} -\Delta v_\epsilon &= \xi_\epsilon v_\epsilon^{p_\epsilon - 1} & \text{in } \ \Omega_\epsilon \\ \frac{\partial v_\epsilon}{\partial \nu} &= -Z(\int_{\partial B_1} u_\epsilon^{q_\epsilon})^{2/q_\epsilon - 1} v_\epsilon^{q_\epsilon - 1} - \frac{n-2}{2}\mu_\epsilon v_\epsilon & \text{on } \ \partial\Omega_\epsilon. \end{cases} \tag{1.14}$$

Lemma 1.3

$$\lim_{\epsilon\to 0}\int_{\partial B_1} u_\epsilon^{q_\epsilon} > 0.$$

Proof. We prove this lemma by contradiction. If not, there exists a subsequence of ϵ (we still denote it as ϵ), s.t.

$$\int_{\partial B_1} u_\epsilon^{q_\epsilon} \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{1.15}$$

From Lemma 1.1 we know: as ϵ small enough, $0 \le \xi_\epsilon \le \xi_Z < 1/S_1$, thus $||u_\epsilon||_{H^1(B_1)} \le C$ uniformly for some constant C. It follows that there exists a $u_0 \in H^1(B_1)$ such that (after passing to a subsequence of ϵ)

$$u_\epsilon \to u_0 \;\text{ weakly in }\; H^1(B_1).$$

Using Corollary 1.2, we know for some small constant δ_0 satisfying $1/(S_1+\delta_0) > \xi_Z$ that

$$\begin{aligned} \xi_\epsilon &= \textstyle\int_{B_1} |\nabla u_\epsilon|^2 + Z(\int_{\partial B_1} u_\epsilon^{q_\epsilon})^{2/q_\epsilon} + \frac{n-2}{2}\int_{\partial B_1} u_\epsilon^2 \\ &\ge \textstyle\frac{1}{S_1+\delta_0}(\int_{B_1} |u_\epsilon|^{p_\epsilon})^{2/p_\epsilon} + o_\epsilon(1) \\ &= \textstyle\frac{1}{S_1+\delta_0} + o_\epsilon(1) \end{aligned}$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$. We derive a contradiction when ϵ is small enough. Lemma 1.3 is established.

Set

$$T = \lim_{\epsilon\to\infty} \frac{dist(x_\epsilon, \partial B_1)}{\mu_\epsilon}. \tag{1.16}$$

(Recall the notation at the end of the introduction, we define the above limit by passing to a subsequence of ϵ.)

Lemma 1.4

$$T < \infty.$$

Proof. If $T = \infty$, we know that $v_\epsilon \to v_1$ in $C^2(B_R(0))$ for any $R > 1$, where v_1 satisfies

$$\begin{cases} -\Delta v_1 &= \xi_0 v_1^{p-1} \quad \text{in } \mathbb{R}^n \\ v_1(0) &= 1, \qquad 0 \le v_1 \le 1 \end{cases}$$

and $\xi_0 = \lim_{\epsilon\to 0}\xi_\epsilon$ (recall our notation: we take this limit up to some subsequence of ϵ). From [7], we know that $v_1(x) = O(\frac{1}{|x|^{n-2}})$ as $|x| \to \infty$ and $v_1(x) \in D^{1,2}(\mathbb{R}^n)$. Multiplying the above equation by v_1, we have

$$\int_{\mathbb{R}^n} |\nabla v_1|^2 = \xi_0 \int_{\mathbb{R}^n} v_1^p.$$

Also by the definition of S_1, we know

$$\int_{\mathbb{R}^n} |\nabla v_1|^2 \geq \frac{1}{S_1}(\int_{\mathbb{R}^n} v_1^p)^{2/p}.$$

Notice $\xi_0 < 1/S_1$. We conclude that

$$\int_{\mathbb{R}^n} v_1^p > 1.$$

On the other hand, note $||v_\epsilon||_{p_\epsilon,\Omega_\epsilon} = \mu_\epsilon^{2/(p_\epsilon-2)-n/p_\epsilon}||u_\epsilon||_{p_\epsilon,\Omega} = \mu_\epsilon^{(n-2)\epsilon/(p_\epsilon(p_\epsilon-2))} \leq 1$ and $v_\epsilon \to v_1$ in $C^2(B_R(0))$, we have $\int_{\mathbb{R}^n} v_1^p \leq 1$. Contradiction. We complete the proof of Lemma 1.4.

Due to $0 \leq \xi_0 \leq C$, Lemma 1.3 and the fact that Ω_ϵ tends to $\mathbb{R}^n_T = \{x = (x', x_n) : x_n > -T\}$, we know by the standard elliptic estimates that up to a subsequence, $v_\epsilon \to v$ in $C^2(B_R \cap \mathbb{R}^n_T)$, and v satisfies

$$\begin{cases} -\Delta v &= \xi_0 v^{p-1}, \quad 0 \leq v \leq 1 \qquad \text{in } \mathbb{R}^n_T \\ \frac{\partial v}{\partial \nu} &= -Zav^{q-1} \qquad\qquad\qquad \text{on } \partial\mathbb{R}^n_T, \end{cases} \tag{1.17}$$

where $a = \lim_{\epsilon\to 0}(\int_{\partial B_1} u_\epsilon^{q_\epsilon})^{2/q_\epsilon - 1}$. From [16] we know that $v(x) = O(\frac{1}{|x|^{n-2}})$ as $|x| \to \infty$ and $v(x) \in D^{1,2}(\mathbb{R}^n_T)$. Therefore, multiplying (1.17) by v, we have

$$\frac{\int_{\mathbb{R}^n_T} |\nabla v|^2 + Za\int_{\partial\mathbb{R}^n_T} v^q}{\int_{\mathbb{R}^n_T} v^p} = \xi_0.$$

Using rescaling, we know

$$(\int_{\partial B_1} u_\epsilon^{q_\epsilon})^{2/q_\epsilon - 1} = \mu_\epsilon^{(\frac{2q_\epsilon}{p_\epsilon - 2} - (n-1))\cdot(1-\frac{2}{q_\epsilon})}(\int_{\partial\Omega_\epsilon} v_\epsilon^{q_\epsilon})^{2/q_\epsilon - 1}. \tag{1.18}$$

Since $(\frac{2q_\epsilon}{p_\epsilon - 2} - (n-1))\cdot(1-\frac{2}{q_\epsilon}) > 0$ and $\mu_\epsilon \to 0$, we know

$$\begin{aligned}(\textstyle\int_{\partial\mathbb{R}^n_T} v^q)^{2/q-1} &\geq \lim_{\epsilon\to 0}(\textstyle\int_{\partial\Omega_\epsilon} v_\epsilon^{q_\epsilon})^{2/q_\epsilon - 1} \\ &\geq a.\end{aligned}$$

If $Z \leq 0$, noticing $||v||_{p,\mathbb{R}^n_T} \leq 1$, we have

$$\begin{aligned} \frac{\int_{\mathbb{R}^n_T} |\nabla v|^2 + Z(\int_{\partial\mathbb{R}^n_T} v^q)^{2/q}}{(\int_{\mathbb{R}^n_T} v^p)^{2/p}} &\leq \frac{\int_{\mathbb{R}^n_T} |\nabla v|^2 + Z\cdot a\cdot \int_{\partial\mathbb{R}^n_T} v^q}{\int_{\mathbb{R}^n_T} v^p} \\ &= \xi_0 \leq \xi_Z. \end{aligned}$$

Set $\bar{v}(x', x_n) = v(x', x_n - T)$, we know that $\inf I_Z$ is attained by $\bar{v}$, contradiction. Thus we complete the proof of Proposition 1.1 in the case of $Z \leq 0$.

Now we consider the case $Z > 0$. Using rescaling, we have

$$\begin{aligned} (\int_{\mathbb{R}^n_T} v^p)^{1-\frac{2}{p}} &\leq \lim_{\epsilon\to 0}(\int_{\Omega_\epsilon} v_\epsilon^{p_\epsilon})^{1-\frac{2}{p_\epsilon}} \\ &\leq \lim_{\epsilon\to 0} \mu_\epsilon^{\frac{(n-2)\epsilon}{p_\epsilon}}. \end{aligned} \tag{1.19}$$

Let

$$a_1 = \lim_{\epsilon\to 0} \mu_\epsilon^{(\frac{2q_\epsilon}{p_\epsilon - 2} - (n-1))\cdot(1-\frac{2}{q_\epsilon})}, \quad b = ||v||^{p-2}_{p,\mathbb{R}^n_T}.$$

Note as ϵ small enough

$$\frac{(n-2)\epsilon}{p_\epsilon} > (\frac{2q_\epsilon}{p_\epsilon - 2} - (n-1)) \cdot (1 - \frac{2}{q_\epsilon}) > 0, \tag{1.20}$$

we know $b \leq a_1 \leq 1$.

In order to complete the proof of Proposition 1.1, we still need one more lemma.

Lemma 1.5

$$\int_{\partial\Omega_\epsilon} v_\epsilon^{q_\epsilon} \to \int_{\partial\mathbb{R}^n_T} v^q \quad as \quad \epsilon \to 0. \tag{1.21}$$

We relegate the proof of this lemma at the end of this Section and continue our proof of Proposition 1.1.

It follows from the above lemma and (1.18) that $a = a_1 ||v||^{2-q}_{q,\partial\mathbb{R}^n_T}$. Therefore

$$\begin{aligned} \frac{\int_{\mathbb{R}^n_T} |\nabla v|^2 + Z(\int_{\partial\mathbb{R}^n_T} v^q)^{2/q}}{(\int_{\mathbb{R}^n_T} v^p)^{2/p}} &\leq \frac{\int_{\mathbb{R}^n_T} |\nabla v|^2}{\int_{\mathbb{R}^n_T} v^p} + \frac{Z\cdot b\cdot(\int_{\partial\mathbb{R}^n_T} v^q)^{2/q}}{\int_{\mathbb{R}^n_T} v^p} \\ &\leq \frac{\int_{\mathbb{R}^n_T} |\nabla v|^2 + Z\cdot a\cdot \int_{\partial\mathbb{R}^n_T} v^q}{\int_{\mathbb{R}^n_T} v^p} \\ &= \xi_0 \leq \xi_Z. \end{aligned}$$

Set $\bar{v}(x', x_n) = v(x', x_n - T)$, we know that $\inf I_Z$ is attained by $\bar{v}$, contradiction. This completes the proof of Proposition 1.1.

Now we are ready to give the proof of Theorem 0.1.

Proof of Theorem 0.1. We first claim: if $Z < Z_0$, $\inf I_Z < 1/S_1$.

For any $d > 0$, we define

$$v_d(x) = (\frac{1}{1+|x'|^2+(x_n-d)^2})^{\frac{n-2}{2}}.$$

Direct calculation shows

$$(\int_{\partial\mathbb{R}^n_+} v_d^q)^{2/q} = C_n^{\frac{n-2}{n-1}}(1+d^2)^{-\frac{n-2}{2}}, \quad (\int_{\mathbb{R}^n_+} v_d^p)^{2/p} = \Big(D_nE_n - D_n\int_d^\infty (1+t^2)^{-\frac{n+1}{2}}dt\Big)^{2/p},$$

and

$$\begin{aligned} \textstyle\int_{\mathbb{R}^n_+}|\nabla v_d|^2 &= (n-2)^2C_nF_n - (n-2)^2C_n\textstyle\int_d^\infty(1+t^2)^{-\frac{n-1}{2}}dt \\ &\quad -(n-2)^2D_nE_n + (n-2)^2D_n\textstyle\int_d^\infty(1+t^2)^{-\frac{n+1}{2}}dt, \end{aligned}$$

where

$$C_n = \int_{\mathbb{R}^{n-1}}(\frac{1}{1+|x'|^2})^{n-1}dx', \quad D_n = \int_{\mathbb{R}^{n-1}}(\frac{1}{1+|x'|^2})^{n}dx',$$

$$E_n = \int_{-\infty}^{+\infty}(1+t^2)^{-\frac{n+1}{2}}dt, \quad F_n = \int_{-\infty}^{+\infty}(1+t^2)^{-\frac{n-1}{2}}dt$$

satisfying the relation

$$\frac{(n-2)^2C_nF_n - (n-2)^2D_nE_n}{(D_nE_n)^{2/p}} = \frac{1}{S_1}.$$

It follows that as d large enough,

$$I_Z(v_d) = \frac{1}{S_1} + \frac{1}{(D_nE_n)^{2/p}}\cdot(ZC_n^{\frac{n-2}{n-1}} - (n-2)C_n)d^{-(n-2)} + o(1)d^{-(n-2)}, \tag{1.22}$$

where $o(1) \to 0$ as $d \to \infty$. Note $Z < Z_0$, $ZC_n^{\frac{n-2}{n-1}} - (n-2)C_n < 0$. Choosing d sufficiently large we establish the claim.

Due to Proposition 1.1, we know that for $-1/S < Z < Z_0$, $\inf I_Z$ is attained by some $u_z \in D^{1,2}(\mathbb{R}^n_+)$. Without loss of generality we can assume that u_z satisfies $||u_z||_{p,\mathbb{R}^n_+} = 1$ and $u_z \geq 0$. Then one can easily see that u_z satisfies the following equation:

$$\begin{cases} -\Delta u_z = \xi_Z u_z^{p-1} & \text{in } \mathbb{R}^n_+ \\ \frac{\partial u_z}{\partial\nu} = -Z(\int_{\partial\mathbb{R}^n_+} u_z^q)^{2/q-1}u_z^{q-1} & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{1.23}$$

Let $v_z = k_z u_z$ with $k_z = (\frac{\xi_Z}{n(n-2)})^{(n-2)/4}$. Then v_z satisfies:

$$\begin{cases} -\Delta v_z = n(n-2)v_1^{p-1} & \text{in } \mathbb{R}^n_+ \\ \frac{\partial v_z}{\partial \nu} = -Z(\int_{\partial \mathbb{R}^n_+} v_z^q)^{2/q-1} v_z^{q-1} & \text{on } \partial \mathbb{R}^n_+. \end{cases} \tag{1.24}$$

From [16] we know

$$v_z = (\frac{\epsilon_z}{\epsilon_z^2 + |x'|^2 + (t-t_z)^2})^{\frac{n-2}{2}},$$

where

$$t_z = (n-2)^{-1}\epsilon_z Z(\int_{\partial \mathbb{R}^n_+} v_z^q)^{2/q-1}$$

for some $\epsilon_z > 0$. Direct computation yields

$$(\int_{\partial \mathbb{R}^n_+} v_z^q)^{2/q-1} = C_n^{-\frac{1}{n-1}}(1 + (\frac{t_z}{\epsilon_z})^2)^{1/2}.$$

Combining the above two identities together, we have

$$Z = \frac{(n-2)C_n^{\frac{1}{n-1}} \frac{t_z}{\epsilon_z}}{(1 + (\frac{t_z}{\epsilon_z})^2)^{1/2}}.$$

This gives the proof of Theorem 0.1 for $-1/S < Z < Z_0$.

If $Z = Z_0$, we will show that

$$\inf I_{Z_0}(u) = \frac{1}{S_1}$$

and the infimum can not be attained through an argument by contradiction. It is well known that $\inf I_{Z_0}(u) \le 1/S_1$. If $\inf I_{Z_0}(u) < 1/S_1$, due to Proposition 1.1, we know that there exists a $u_0 \ge 0$ with $||u_0||_{p,\mathbb{R}^n_+} = 1$ such that $I_{Z_0}(u_0) = \inf I_{Z_0}(u) := \xi_{Z_0}$. It follows that u_0 satisfies

$$\begin{cases} -\Delta u_0 = \xi_{Z_0} u_0^{p-1} & \text{in } \mathbb{R}^n_+ \\ \frac{\partial u_0}{\partial \nu} = -Z_0(\int_{\partial \mathbb{R}^n_+} u_0^q)^{2/q-1} u_0^{q-1} & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$

It can be shown as the above that $u_0 = (\xi_{Z_0}/(n(n-2)))^{-(n-2)/4} v_{Z_0}$, where

$$v_{Z_0} = (\frac{\epsilon_0}{\epsilon_0^2 + |x'|^2 + (t-t_0)^2})^{\frac{n-2}{2}}$$

with

$$t_0 = (n-2)^{-1}\epsilon_0 Z_0 (\int_{\partial\mathbb{R}^n_+} v_{Z_0}^q)^{2/q-1}$$

for some $\epsilon_0 > 0$. Therefore

$$Z_0 = \frac{(n-2)C_n^{\frac{1}{n-1}}\frac{t_0}{\epsilon_0}}{(1+(\frac{t_0}{\epsilon_0})^2)^{1/2}} < (n-2)C_n^{\frac{1}{n-1}} = Z_0.$$

Contradiction! The nonexistence of the extremal functions also follows from the same argument. This completes the proof of Theorem 0.1.

We are left to prove Lemma 1.5. In order to prove Lemma 1.5, we need the following inequality. One can find a proof of such inequality in Adam's book [1](with a slight modification).

Lemma 1.6 *Let Ω be a bounded domain in $\mathbb{R}^n$. For all $\gamma > 0, \epsilon > 0$, there exists a constant $\hat{C}(\gamma,\epsilon)$ depending on γ and ϵ such that*

$$(\int_{\partial\Omega} |u|^{q_\epsilon} ds)^{\frac{2}{q_\epsilon}} \le \gamma \int_\Omega |\nabla u|^2 dv + \hat{C}(\gamma,\epsilon)(\int_\Omega |u|^{p_\epsilon} dv)^{\frac{2}{p_\epsilon}}, \quad \forall\ u \in H^1(\Omega). \tag{1.25}$$

Further, if ϵ is close to 0, one can choose $\hat{C}(\gamma,\epsilon)$ independent of ϵ.

Proof of Lemma 1.5. Since $||u_\epsilon||_{H^1} < C$, we know that $u_\epsilon \to u_0$ weakly in $H^1(B_1)$ for some $u_0 \ge 0$. Noticing $\lim_{\epsilon\to 0}(\int_{\partial B_1} u_\epsilon^{q_\epsilon})^{2/q_\epsilon - 1} = a > 0$, we know that u_0 satisfies

$$\begin{cases} -\Delta u_0 &= \xi_0 u_0^{p-1}, \ \ 0 \le u_0 \le 1, \quad \text{in } B_1 \\ \frac{\partial u_0}{\partial \nu} &= -Zau_0^{q-1} - \frac{n-2}{2}u_0 \quad \text{on } \partial B_1. \end{cases} \tag{1.26}$$

Set $u_{\epsilon,1} = u_\epsilon - u_0$, then $u_{\epsilon,1} \to 0$ weakly in $H^1(B_1)$. Since $\inf II_Z$ is not attained, $||u_{\epsilon,1}||_\infty \to \infty$. Let $x_{\epsilon,1}$ be the maximal point of $u_{\epsilon,1}$, $\mu_{\epsilon,1} = u_{\epsilon,1}(x_{\epsilon,1})^{(p_\epsilon-2)/2}$ and

$$v_{\epsilon,1}(x) = \mu_{\epsilon,1}^{\frac{2}{p_\epsilon-2}} u_{\epsilon,1}(\mu_{\epsilon,1}x + x_{\epsilon,1}) \text{ for } \ x \in \Omega_{\epsilon,1} = \frac{B_1 - x_{\epsilon,1}}{\mu_{\epsilon,1}}.$$

As before, we know, that up to a subsequence, $v_{\epsilon,1} \to v_1$ in $C^2(B_R \cap \mathbb{R}^n_{T_1})$, and v_1 satisfies

$$\begin{cases} -\Delta v_1 &= \xi_0 {v_1}^{p-1}, \ \ 0 \le v_1 \le 1, \ \ v_1(0) = 1, \quad \text{in } \mathbb{R}^n_{T_1} \\ \frac{\partial v_1}{\partial \nu} &= -Za{v_1}^{q-1} \quad \text{on } \partial\mathbb{R}^n_{T_1}, \end{cases} \tag{1.27}$$

where

$$T_1 = \overline{\lim_{\epsilon\to\infty}} \frac{dist(x_{\epsilon,1}, \partial B_1)}{\mu_{\epsilon,1}}.$$

For any $R >> 1$, define

$$u_\epsilon^{(1)} = \mu_{\epsilon,1}^{-\frac{2}{p_\epsilon-2}} v_1(\mu_{\epsilon,1}^{-1}(x - x_{\epsilon,1})) \cdot \eta(\mu_{\epsilon,1}^{-1}(x - x_{\epsilon,1})), \quad x \in B_1,$$

where $\eta(x)$ is a cutoff function with $\eta(x) = 1$ for $x \in B_R(0)$ and $\eta(x) = 0$ in $B_{2R}^c(0)$.

Due to (1.18) and (1.19), we only need to show that $u_0 = 0$ (therefore $u_{\epsilon,1} = u_\epsilon$, $x_{\epsilon,1} = x_\epsilon$ and $T_1 = T$) and

$$||u_{\epsilon,1} - u_\epsilon^{(1)}||_{p_\epsilon,B_1}, \quad ||u_{\epsilon,1} - u_\epsilon^{(1)}||_{q_\epsilon,\partial B_1} = o_\epsilon(1) + o_R(1) \tag{1.28}$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$ and $o_R(1) \to 0$ as $R \to \infty$.

It is easy to see that $||u_{\epsilon,1}||_{p_\epsilon,B_1} \geq C_0 > 0$ as ϵ small enough. Suppose that there exists some $\delta_0 > 0$, such that

$$||u_{\epsilon,1} - u_\epsilon^{(1)}||_{p_\epsilon,B_1} > \delta_0. \tag{1.29}$$

Then we define $u_{\epsilon,2} = u_\epsilon - u_0 - u_\epsilon^{(1)}$. Easy to see that $u_{\epsilon,2} \to 0$ weakly in $H^1(B_1)$. Since $||u_{\epsilon,2}||_{p_\epsilon,B_1} \geq \delta_0$, we know $||u_{\epsilon,2}||_\infty \to \infty$. Let $x_{\epsilon,2}$ be the maximal point of $u_{\epsilon,2}$, $\mu_{\epsilon,2} = u_{\epsilon,2}(x_{\epsilon,2})^{(p_\epsilon-2)/2}$ and

$$v_{\epsilon,2}(x) = \mu_{\epsilon,2}^{\frac{2}{p_\epsilon-2}} u_{\epsilon,2}(\mu_{\epsilon,2}x + x_{\epsilon,2}) \text{ for } \ x \in \Omega_{\epsilon,2} = \frac{B_1 - x_{\epsilon,2}}{\mu_{\epsilon,2}}.$$

One observes that $dist(x_{\epsilon,1}, x_{\epsilon,2}) > 100R\mu_{\epsilon,1}$ for any fixed R as $\epsilon \to 0$. Also $\mu_{\epsilon,1} \leq \mu_{\epsilon,2}$. Therefore, we know as before that, up to a subsequence, $v_{\epsilon,2} \to v_2$ in $C^2(B_R \cap \mathbb{R}^n_{T_2})$, and v_2 satisfies

$$\begin{cases} -\Delta v_2 & = \xi_0 {v_2}^{p-1}, \quad 0 \leq v_2 \leq 1, \quad v_2(0) = 1, \qquad \text{in } \ \mathbb{R}^n_{T_2} \\ \frac{\partial v_2}{\partial \nu} & = -Za{v_2}^{q-1} \qquad\qquad\qquad\qquad\qquad\quad \text{on } \ \partial\mathbb{R}^n_{T_2}, \end{cases} \tag{1.30}$$

where

$$T_2 = \overline{\lim_{\epsilon\to\infty}} \frac{dist(x_{\epsilon,2}, \partial B_1)}{\mu_{\epsilon,2}}.$$

Define

$$u_\epsilon^{(2)} = \mu_{\epsilon,2}^{-\frac{2}{p_\epsilon-2}} v_2(\mu_{\epsilon,2}^{-1}(x - x_{\epsilon,2}))\eta(\mu_{\epsilon,2}^{-1}(x - x_{\epsilon,2})), \quad x \in B_1,$$

where $\eta(x)$ is the cutoff function as the above ($\eta(x) = 1$ for $x \in B_R(0)$ and $\eta(x) = 0$ in $B_{2R}^c(0)$).

Without loss of generality, we can assume that for any $0 < \delta << 1$, as $\epsilon \to 0$

$$||u_{\epsilon,2} - u_\epsilon^{(2)}||_{p_\epsilon, B_1} \leq \delta. \tag{1.31}$$

Otherwise we just keep this process going. Since $||u_\epsilon||_{p_\epsilon,\ B_1} = 1$, we know this process must stop after several steps (depends on δ).

It is easy to check that

$$||u_0 + u_\epsilon^{(1)} + u_\epsilon^{(2)}||^{q_\epsilon}_{q_\epsilon, \partial B_1} = ||u_0||^{q_\epsilon}_{q_\epsilon, \partial B_1} + ||u_\epsilon^{(1)}||^{q_\epsilon}_{q_\epsilon, \partial B_1} + ||u_\epsilon^{(2)}||^{q_\epsilon}_{q_\epsilon, \partial B_1} + o_\epsilon(1) + o_R(1),$$

and

$$||u_0 + u_\epsilon^{(1)} + u_\epsilon^{(2)}||^{p_\epsilon}_{p_\epsilon, B_1} = ||u_0||^{p_\epsilon}_{p_\epsilon, B_1} + ||u_\epsilon^{(1)}||^{p_\epsilon}_{p_\epsilon, B_1} + ||u_\epsilon^{(2)}||^{p_\epsilon}_{p_\epsilon, B_1} + o_\epsilon(1) + o_R(1).$$

Combining with (1.31) and using Lemma 1.6, we have

$$1 - \delta - o_\epsilon(1) - o_R(1) \leq ||u_0||^{p_\epsilon}_{p_\epsilon, B_1} + ||u_\epsilon^{(1)}||^{p_\epsilon}_{p_\epsilon, B_1} + ||u_\epsilon^{(2)}||^{p_\epsilon}_{p_\epsilon, B_1} \leq 1 + \delta + o_\epsilon(1) + o_R(1)$$

and

$$a^{\frac{q}{2-q}} - c(\delta) - o_\epsilon(1) - o_R(1) \leq ||u_0||^{q_\epsilon}_{q_\epsilon, \partial B_1} + ||u_\epsilon^{(1)}||^{q_\epsilon}_{q_\epsilon, \partial B_1} + ||u_\epsilon^{(2)}||^{q_\epsilon}_{q_\epsilon, \partial B_1} \leq a^{\frac{q}{2-q}} + c(\delta) + o_\epsilon(1) + o_R(1)$$

where $c(\delta) \to 0$ as $\delta \to 0$.

Define

$$\alpha_0 = ||u_0||^{p_\epsilon}_{p_\epsilon}, \quad \beta_0 = ||u_0||^{q_\epsilon}_{q_\epsilon} / ||u_\epsilon||^{q_\epsilon}_{q_\epsilon},$$

and

$$\alpha_i = ||u_\epsilon^{(i)}||^{p_\epsilon}_{p_\epsilon}, \quad \beta_i = ||u_\epsilon^{(i)}||^{q_\epsilon}_{q_\epsilon} / ||u_\epsilon||^{q_\epsilon}_{q_\epsilon} \ \text{ for } i = 1, 2.$$

Therefore

$$1 - \delta - o_\epsilon(1) - o_R(1) \leq \alpha_0 + \alpha_1 + \alpha_2 \leq 1 + \delta + o_\epsilon(1) + o_R(1) \tag{1.32}$$

$$1 - c'(\delta) - o_\epsilon(1) - o_R(1) \leq \beta_0 + \beta_1 + \beta_2 \leq 1 + c'(\delta) + o_\epsilon(1) + o_R(1) \tag{1.33}$$

where $c'(\delta) \to 0$ as $\delta \to 0$.

It follows from (1.26), (1.27) and (1.30) that

$$||\nabla u_0||^2_{2, B_1} + Za||u_0||^q_{q, \partial B_1} = \xi_0 ||u_0||^p_{p, B_1},$$

$$||\nabla v_i||^2_{2, \mathbb{R}^+_{T_i}} + Za||v_i||^q_{q, \partial \mathbb{R}^+_{T_i}} = \xi_0 ||v_i||^p_{p, \mathbb{R}^+_{T_i}} \quad \text{for} \quad i = 1, 2.$$

Using (1.18), (1.19), (1.20) and the above, we have

$$||\nabla u_0||^2_{2,B_1} + Z\beta_0^{1-\frac{2}{q}}||u_0||^2_{q,\partial B_1} = \xi_0\alpha_0^{1-\frac{2}{p}}||u_0||^2_{p,B_1} + o_\epsilon(1),$$

$$||\nabla v_i||^2_{2,\mathbb{R}^+_{T_i}} + Z\beta_i^{1-\frac{2}{q}}||v_i||^2_{q,\partial\mathbb{R}^+_{T_i}} \leq \xi_0\alpha_i^{1-\frac{2}{p}}||v_i||^2_{p,\mathbb{R}^+_{T_i}} + o_\epsilon(1) + o_R(1) \quad \text{for} \quad i = 1, 2.$$

Since $\inf I_Z \geq \xi_0$ and $\alpha_i \leq 1$, we know that as $\epsilon \to 0$ and $R \to \infty$,

$$\beta_i^{1-\frac{2}{q}} \leq \alpha_i^{1-\frac{2}{p}} \quad \text{for} \quad i = 0, 1, 2. \tag{1.34}$$

Noticing $q < p$ and α_1, α_2 are larger than some fixed number, we derive a contradiction due to (1.32), (1.33) and (1.34) when we choose δ suitable small and R suitable large. Thus (1.29) is false, that is: as $\epsilon \to 0$,

$$||u_{\epsilon,1} - u_\epsilon^{(1)}||_{p_\epsilon,B_1} \to 0.$$

Using Lemma 1.6, we know that (1.28) holds. Similarly, we can show that $u_0 = 0$. Lemma 1.5 is established.

2 Domain case

In this section, we assume that Ω is a smooth bounded domain in $\mathbb{R}^n$ and give the proof of Theorem 0.2.

First we present a rough inequality with a slight larger constant than the sharp one $S_1(Z)$ in Theorem 0.2. However, the case $Z = Z_0$ is included.

Proposition 2.1 *Let $Z \in (-1/S, Z_0]$. For any $\delta > 0$, there exists $C(\delta) > 0$ such that*

$$(\int_\Omega |u|^p)^{\frac{2}{p}} \leq (S_1(Z)+\delta)\Big(\int_\Omega |\nabla u|^2 + Z(\int_{\partial\Omega} |u|^q)^{\frac{2}{q}}\Big) + C(\delta)\int_{\partial\Omega} u^2, \ \ \forall u \in H^1(\Omega). \tag{2.1}$$

When $Z > 0$, due to the positive L^q term in the right hand side of (2.1), we can not prove this inequality directly from Theorem 0.1 via the partition of unit, neither can we prove it by a similar argument used in [17] and [18]. Here, we again use blowup argument to prove this proposition.

Proof. We prove it by contradiction. Assume (2.1) is not true, that is, there exists $\delta_1 > 0$ such that $\forall \alpha > 1$,

$$\inf_{u \in H^1(\Omega)} I_{Z,\alpha}(u) := \inf_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 + Z(\int_{\partial\Omega} |u|^q)^{2/q} + \alpha \int_{\partial\Omega} u^2}{(\int_\Omega |u|^p)^{2/p}} := \xi_{Z,\alpha} < \frac{1}{S_1(Z) + \delta_1}. \tag{2.2}$$

Let $\bar{\alpha}$ be some positive constant such that

$$\frac{1}{S} ||u||^2_{q,\partial\Omega} \leq ||\nabla u||^2_{2,\Omega} + \bar{\alpha} ||u||^2_{2,\partial\Omega}, \quad \forall u \in H^1(\Omega). \tag{2.3}$$

The existence of such $\bar{\alpha}$ was shown in [17].

Lemma 2.1 *If $Z \geq 0$, under condition (2.2),* $\inf I_{Z,\alpha}$ *is achieved; If $-1/S < Z < 0$, for any fixed $\alpha > \bar{\alpha}$, under condition (2.2),* $\inf I_{Z,\alpha}$ *is achieved.*

Proof. This can be proved in a similar way as that of Proposition 1.1. We sketch the proof here for readers' convenience. For $u \neq 0$, define

$$I_\epsilon(u) = \frac{\int_\Omega |\nabla u|^2 + Z(\int_{\partial\Omega} |u|^{q_\epsilon})^{2/q_\epsilon} + \alpha \int_{\partial\Omega} u^2}{(\int_\Omega |u|^{p_\epsilon})^{2/p_\epsilon}}.$$

If $Z \geq 0$, or $-1/S < Z < 0$ and $\alpha > \bar{\alpha}$, as before, we know $I_\epsilon(u) \geq 0$. The standard variational method shows that $\exists\ u_\epsilon \geq 0$ with $||u_\epsilon||_{p_\epsilon} = 1$ such that

$$I_\epsilon(u_\epsilon) = \inf I_\epsilon(u) := \xi_\epsilon.$$

We want to show that $||u_\epsilon||_\infty \leq C$. Lemma 2.1 follows from this fact easily.

Suppose $||u_\epsilon||_\infty \to \infty$ up to a subsequence. Easy to see that u_ϵ satisfies

$$\begin{cases} -\Delta u_\epsilon &= \xi_\epsilon u_\epsilon^{p_\epsilon - 1} & \text{in } \Omega \\ \frac{\partial u_\epsilon}{\partial \nu} &= -Z(\int_{\partial\Omega} u_\epsilon^{q_\epsilon})^{2/q_\epsilon - 1} u_\epsilon^{q_\epsilon - 1} - \alpha u_\epsilon & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

By [10], we know that there exists a $x_\epsilon \in \Omega$ such that $u_\epsilon(x_\epsilon) = ||u_\epsilon||_\infty \to \infty$. Define

$$\begin{cases} \mu_\epsilon &= (v_\epsilon(x_\epsilon))^{-\frac{p_\epsilon - 2}{2}}, \quad \Omega_\epsilon = \frac{\Omega - x_\epsilon}{\mu_\epsilon}, \\ v_\epsilon(x) &= \mu_\epsilon^{\frac{2}{p_\epsilon - 2}} u_\epsilon(\mu_\epsilon x + x_\epsilon) \ \text{ for } \ x \in \Omega_\epsilon. \end{cases}$$

Then v_ϵ satisfies

$$\begin{cases} -\Delta v_\epsilon &= \xi_\epsilon v_\epsilon^{p_\epsilon - 1}, \quad 0 \leq v_\epsilon \leq 1, \quad v(0) = 1, & \text{in } \Omega_\epsilon \\ \frac{\partial v_\epsilon}{\partial \nu} &= -Z(\int_{\partial\Omega} u_\epsilon^{q_\epsilon})^{2/q_\epsilon - 1} v_\epsilon^{q_\epsilon - 1} - \alpha \mu_\epsilon v_\epsilon & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{2.5}$$

As in the proof of Proposition 1.1 (also we need to use (2.3) when $Z < 0$), we can show that when $Z \geq 0$, or $-1/S < Z < 0$ and $\alpha > \bar{\alpha}$,

$$0 \leq \lim_{\epsilon \to 0} \xi_\epsilon := \xi_0 \leq \xi_{Z,\alpha} < \frac{1}{S_1(Z) + \delta_1} \tag{2.6}$$

and

$$\int_{\partial\Omega} u_\epsilon^{q_\epsilon} \geq C, \quad T := \overline{\lim_{\epsilon \to \infty}} \frac{dist(x_\epsilon, \partial\Omega)}{\mu_\epsilon} < \infty. \tag{2.7}$$

It is easy to see $\alpha||u_\epsilon||_{2,\partial\Omega} \leq C$. Combining this with (2.7) and the definition of μ_ϵ, we have

$$\alpha\mu_\alpha \leq \frac{\alpha\mu_\alpha}{C} \int_{\partial\Omega} u_\epsilon^{q_\epsilon} \leq \frac{\alpha}{C} \int_{\partial\Omega} u_\epsilon^2 \leq C. \tag{2.8}$$

Set $C_1 = \lim_{\epsilon \to 0} \alpha\mu_\epsilon$. By standard elliptic estimates, from (2.5), (2.7) and (2.8), we know that $v_\epsilon \to v$ in $C^2(B_R \cap \mathbb{R}^n_T)$, and v satisfies

$$\begin{cases} -\Delta v &= \xi_0 v^{p-1}, \quad 0 \leq v \leq 1, \quad v(0) = 1, \qquad \text{in } \mathbb{R}^n_T \\ \frac{\partial v}{\partial \nu} &= -Zav^{q-1} - C_1 v \qquad\qquad\qquad\qquad\quad \text{on } \partial\mathbb{R}^n_T. \end{cases} \tag{2.9}$$

where $a = \lim_{\epsilon \to 0}(\int_{\partial\Omega} u_\epsilon^{q_\epsilon})^{2/q_\epsilon - 1}$. If $Z \leq 0$, notice that $||v||_{p,\mathbb{R}^n_T} \leq 1$ and $(\int_{\partial\mathbb{R}^n_T} v^q)^{2/q-1} \geq a$ (see (1.18) for details), we have

$$\begin{aligned} &\frac{\int_{\mathbb{R}^n_T} |\nabla v|^2 + Z(\int_{\partial\mathbb{R}^n_T} v^q)^{2/q}}{(\int_{\mathbb{R}^n_T} v^p)^{2/p}} \\ \leq &\frac{\int_{\mathbb{R}^n_T} |\nabla v|^2 + Za\int_{\partial\mathbb{R}^n_T} v^q}{\int_{\mathbb{R}^n_T} v^p} \leq \xi_0 < \frac{1}{S_1(Z)}. \end{aligned}$$

This contradicts to Theorem 0.1.

If $Z > 0$, slightly modifying the proof of Lemma 1.5 (we need to use Theorem 0.1 here), we can show that as $\epsilon \to 0$

$$\int_{\partial\Omega_\epsilon} v_\epsilon^{q_\epsilon} \to \int_{\partial\mathbb{R}^n_T} v^q.$$

Then following the proof of Proposition 1.1 closely, we can get

$$\frac{\int_{\mathbb{R}^n_T} |\nabla v|^2 + Z(\int_{\partial\mathbb{R}^n_T} v^q)^{2/q}}{(\int_{\mathbb{R}^n_T} v^p)^{2/p}} \leq \xi_0 < \frac{1}{S_1(Z)}.$$

This again contradicts to Theorem 0.1. We thereby complete the proof of Lemma 2.1.

As $\alpha > \bar{\alpha}$, without loss of generality, we can assume that $\inf I_{Z,\alpha}(u) = I_{Z,\alpha}(u_\alpha)$ with $u_\alpha \geq 0$ and $||u_\alpha||_{p,\Omega} = 1$. It is easy to see that u_α satisfies

$$\begin{cases} -\Delta u_\alpha &= \xi_{Z,\alpha} u_\alpha^{p-1} \qquad \text{in } \Omega \\ \frac{\partial u_\alpha}{\partial \nu} &= -Z(\int_{\partial\Omega} u_\alpha^q)^{2/q-1} u_\alpha^{q-1} - \alpha u_\alpha \qquad \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

and

$$\alpha ||u_\alpha||_{2,\partial\Omega} \leq C. \tag{2.11}$$

As in the proof of Lemma 1.3, due to $\xi_{Z,\alpha} < 1/(S_1 + \delta_1)$, we know that there exists a constant $C > 0$, such that

$$\int_{\partial\Omega} u_\alpha^q \geq C. \tag{2.12}$$

Due to Cherrier, we know that u_α is smooth up to the boundary. Let $u_\alpha(x_\alpha) = ||u_\alpha||_\infty$ for some $x_\alpha \in \Omega$, and define

$$\begin{cases} \mu_\alpha &= (u_\alpha(x_\alpha))^{-\frac{p-2}{2}}, \quad \Omega_\alpha = (\Omega - x_\alpha)/\mu_\alpha, \\ v_\alpha(x) &= \mu_\alpha^{\frac{2}{p-2}} u_\alpha(\mu_\alpha x + x_\alpha), \quad x \in \Omega_\alpha. \end{cases} \tag{2.13}$$

Then v_α satisfies

$$\begin{cases} -\Delta v_\alpha &= \xi_{Z,\alpha} v_\alpha^{p-1}, \quad 0 \leq v_\alpha \leq 1, \quad v(0) = 1, \qquad \text{in } \Omega_\alpha \\ \frac{\partial v_\alpha}{\partial \nu} &= -Z(\int_{\partial\Omega_\alpha} v_\alpha^q)^{2/q-1} v_\alpha^{q-1} - \alpha\mu_\alpha v_\alpha \qquad \text{on } \partial\Omega_\alpha. \end{cases} \tag{2.14}$$

From (2.11), (2.12) and the definition of μ_α, we know that

$$\alpha\mu_\alpha \leq \frac{\alpha\mu_\alpha}{C} \int_{\partial\Omega} u_\alpha^q = \frac{\alpha}{C} \int_{\partial\Omega} u_\alpha^2 \leq C.$$

Set $C_2 = \lim_{\alpha\to\infty} \alpha\mu_\alpha \geq 0$. Thus $||u_\alpha||_\infty \to \infty$. Also as in the proof of Proposition 1.1, due to $\xi_{Z,\alpha} < 1/(S_1 + \delta_1)$, we know

$$\overline{\lim_{\alpha\to\infty}} \frac{dist(x_\alpha, \partial\Omega)}{\mu_\alpha} = T < \infty. \tag{2.15}$$

By standard elliptic estimates, we know that $v_\epsilon \to v$ in $C^2(B_R \cap \mathbb{R}^n_T)$, and $v \neq 0$ satisfies

$$\begin{cases} -\Delta v &= \xi_\infty v^{p-1}, \quad 0 \leq v \leq 1, \quad v(0) = 1, \qquad \text{in } \mathbb{R}^n_T \\ \frac{\partial v}{\partial \nu} &= -Z a_1 v^{q-1} - C_2 v \qquad \text{on } \partial\mathbb{R}^n_T \end{cases} \tag{2.16}$$

where $\xi_\infty = \lim_{\alpha\to\infty} \xi_{Z,\alpha} \leq 1/(S_1(Z) + \delta_1)$, $a_1 = \lim_{\alpha\to\infty} (\int_{\partial\Omega_\alpha} v_\alpha^q)^{2/q-1}$. If $Z \leq 0$, one can easily see as before that

$$\frac{\int_{\mathbb{R}^n_T} |\nabla v|^2 + Z(\int_{\partial\mathbb{R}^n_T} v^q)^{2/q}}{(\int_{\mathbb{R}^n_T} v^p)^{2/p}} \leq \frac{\int_{\mathbb{R}^n_T} |\nabla v|^2 + Z(\int_{\partial\mathbb{R}^n_T} v^q)^{2/q} + C_2 \int_{\partial\mathbb{R}^n_T} v^2}{\int_{\mathbb{R}^n_T} v^p} = \xi_\infty < \frac{1}{S_1(Z)}.$$

If $Z > 0$, similarly, using Theorem 0.1, we can prove as in the proof of Lemma 1.5 that

$$\lim_{\alpha\to\infty}(\int_{\partial\Omega_\alpha} v_\alpha^q)^{2/q-1} = (\int_{\partial\mathbb{R}_T^n} v^q)^{2/q-1},$$

thus we also have

$$\frac{\int_{\mathbb{R}_T^n} |\nabla v|^2 + Z(\int_{\partial\mathbb{R}_T^n} v^q)^{2/q}}{(\int_{\mathbb{R}_T^n} v^p)^{2/p}} \leq \frac{\int_{\mathbb{R}_T^n} |\nabla v|^2 + Z(\int_{\partial\mathbb{R}_T^n} v^q)^{2/q} + C_2 \int_{\partial\mathbb{R}_T^n} v^2}{\int_{\mathbb{R}_T^n} v^p} = \xi_\infty < \frac{1}{S_1(Z)}.$$

Both of the above two inequalities contradict to Theorem 0.1. Thus the proof of Proposition 2.1 is completed.

From now on, we begin to prove Theorem 0.2 through an argument by contradiction. Note that we assume $Z < Z_0$, thus $1/S_1(Z) < 1/S_1$.

Suppose Theorem 0.2 is false, then for any $\alpha > \bar{\alpha}$,

$$\inf_{H^1(\Omega)} I_\alpha(u) = \inf_{H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 + Z(\int_{\partial\Omega} |u|^q)^{2/q} + \alpha \int_{\partial\Omega} u^2}{(\int_\Omega |u|^p)^{2/p}} := \xi_\alpha < \frac{1}{S_1(Z)}. \tag{2.17}$$

From the proof of Proposition 2.1, we know that under assumption (2.17), $\inf I_\alpha(u)$ is attained. Without loss of generality, we assume $\inf I_\alpha(u) = I_\alpha(u_\alpha)$ with $u_\alpha \geq 0$ and $||u_\alpha||_{p,\Omega} = 1$. It is easy to see that u_α satisfies

$$\begin{cases} -\Delta u_\alpha &= \xi_\alpha u_\alpha^{p-1} & \text{in } \Omega \\ \frac{\partial u_\alpha}{\partial \nu} &= -Z(\int_{\partial\Omega} u_\alpha^q)^{2/q-1} u_\alpha^{q-1} - \alpha u_\alpha & \text{on } \partial\Omega. \end{cases} \tag{2.18}$$

Lemma 2.2 *As $\alpha \to \infty$,*

$$\alpha ||u_\alpha||_{2,\partial\Omega}^2 \to 0, \quad \xi_\alpha \to \frac{1}{S_1(Z)}. \tag{2.19}$$

Proof. From (2.17), Proposition 2.1 and $||u_\alpha||_{p,\Omega} = 1$, we know that for any $\delta > 0$, there exists a constant $C(\delta)$ such that

$$\begin{aligned} 1 + \tfrac{\delta}{S_1(Z)} &\geq (S_1(Z) + \delta)\xi_\alpha \\ &= (S_1(Z) + \delta)(||\nabla u_\alpha||_{2,\Omega}^2 + Z||u_\alpha||_{q,\partial\Omega}^2 + \alpha ||u_\alpha||_{2,\partial\Omega}^2) \\ &\geq 1 + (\alpha(S_1(Z) + \delta) - C(\delta))||u_\alpha||_{2,\partial\Omega}^2. \end{aligned}$$

Thus

$$1 + \frac{\delta}{S_1(Z)} \geq (S_1(Z) + \delta) \limsup_{\alpha\to\infty} \xi_\alpha \geq 1 + (S_1(Z) + \delta) \limsup_{\alpha\to\infty} \alpha \|u_\alpha\|_{2,\partial\Omega}^2,$$

$$1+\frac{\delta}{S_1(Z)} \geq (S_1(Z)+\delta)\liminf_{\alpha\to\infty}\xi_\alpha \geq 1+(S_1(Z)+\delta)\liminf_{\alpha\to\infty}\alpha\|u_\alpha\|^2_{2,\partial\Omega}.$$

Sending $\delta \to 0$, we have our lemma.

Since $1/S_1(Z) < 1/S_1$, as in the proof of Lemma 1.3, we have the following.

Lemma 2.3 *There exists a constant $C > 0$ such that*

$$\int_{\partial\Omega} u_\alpha^q \geq C. \tag{2.20}$$

Remark 2.1 *Condition $Z < Z_0$ (thus $1/S_1(Z) < 1/S_1$) is essential in the proof of (2.20). Actually, if $Z = Z_0$, we can show that $\int_{\partial\Omega} u_\alpha^q \to 0$ as $\alpha \to \infty$. More details will be discussed in Section 4.*

Since u_α satisfies (2.18), due to Cherrier, we know that u_α is smooth up to the boundary. Let $u_\alpha(x_\alpha) = ||u||_\infty$ for some $x_\alpha \in \Omega$, and define

$$\begin{cases} \mu_\alpha & = (u_\alpha(x_\alpha))^{-\frac{p-2}{2}}, \quad \Omega_\alpha = (\Omega - x_\alpha)/\mu_\alpha, \\ v_\alpha(x) & = \mu_\alpha^{\frac{2}{p-2}} u_\alpha(\mu_\alpha x + x_\alpha) \ \text{ for } \ x \in \Omega_\alpha. \end{cases} \tag{2.21}$$

Then v_α satisfies

$$\begin{cases} -\Delta v_\alpha & = \xi_\alpha v_\alpha^{p-1}, \quad 0 \leq v \leq 1, \quad v(0) = 1 \qquad \text{in } \ \Omega_\alpha \\ \frac{\partial v_\alpha}{\partial \nu} & = -Z(\int_{\partial\Omega_\alpha} v_\alpha^q)^{2/q-1} v_\alpha^{q-1} - \alpha\mu_\alpha v_\alpha \qquad \text{on } \ \partial\Omega_\alpha. \end{cases} \tag{2.22}$$

Combining (2.19) with (2.20), as being shown in (2.8), we have

$$\alpha\mu_\alpha \to 0. \tag{2.23}$$

Also if we set $\overline{\lim}_{\alpha\to\infty} dist(x_\alpha, \partial\Omega)/\mu_\alpha = T$, due to $1/S_1(Z) < 1/S_1$, as before, we know $T < \infty$.

By standard elliptic estimates we know that $v_\alpha \to v$ in $C^3(\overline{\Omega_\alpha} \cap B_R(0))$ for all $R > 1$. And again, using the argument in the proof of Lemma 1.5 (also we need to use Theorem 0.1 here), we know

$$\lim_{\alpha\to\infty}\int_{\Omega_\alpha}|v_\alpha - v|^p = \lim_{\alpha\to\infty}\int_{\Omega_\alpha}|\nabla v_\alpha - \nabla v|^2 = \lim_{\alpha\to\infty}\int_{\partial\Omega_\alpha}|v_\alpha - v|^q = 0. \tag{2.24}$$

It follows that

$$\lim_{\alpha\to\infty}(\int_{\partial\Omega_\alpha} v_\alpha^q)^{2/q-1} = (\int_{\partial\mathbb{R}^n_T} v^q)^{2/q-1},$$

and v satisfies

$$\begin{cases} -\Delta v = \frac{1}{S_1(Z)} v^{p-1}, \quad 0 \leq v(x) \leq 1, \quad v(0) = 1, & \text{in } \ \mathbb{R}^n_T, \\ \frac{\partial v}{\partial \nu} = -Z(\int_{\partial \mathbb{R}^n_T} v^q)^{2/q-1} v^{q-1} & \text{on } \ \partial \mathbb{R}^n_T. \end{cases} \tag{2.25}$$

If $Z \geq 0$, it follows from [16] that

$$v = (\frac{1}{1 + c(n)(|x'|^2 + |x_n|^2)})^{\frac{n-2}{2}} \tag{2.26}$$

where $c(n) = 1/(S_1(Z)(n-2)n)$.

If $Z < 0$, due to $1/S_1(Z) < 1/(2^{2/n} S_1)$, one can check as in the proof of Lemma 1.4 that $T = 0$. Due to (2.24), we know $||v||_{p, \mathbb{R}^n_+} = 1$. It follows from the proof of Theorem 0.1 that

$$v = (\frac{\epsilon_z}{\epsilon_z^2 + |x'|^2 + (x_n - \epsilon_z d_z)^2})^{\frac{n-2}{2}} \tag{2.27}$$

where d_z satisfies (0.3), $\epsilon_z = (Z_0^2 - Z^2)/Z_0^2$.

We are ready to give a L^∞ estimate on v_α through the Moser iteration method as we did in [17] and [18]. First, Let's recall that the conformal Laplacian operator L_g and the conformal boundary operator B_g corresponding to metric g are given by

$$\begin{cases} L_g \psi = \Delta_g \psi - a(n) R_g \psi, \\ B_g \psi = \frac{\partial_g \psi}{\partial \nu} + b(n) H_g \psi, \end{cases} \tag{2.28}$$

where $a(n) = \frac{n-2}{4(n-1)}$, $b(n) = \frac{n-2}{2}$, R_g is the scalar curvature of Ω, and H_g is the mean curvature of $\partial\Omega$ with respect to the inner normal of $\partial\Omega$ (e.g., the unit ball in $\mathbb{R}^n$ has positive mean curvature).

We write g_0 as the standard Euclidean metric. Let $v(x)$ be given by (2.26) or (2.27), and $\hat{g} = v^{4/(n-2)} g_0$, i.e. $\hat{g}_{ij} dx^i dx^j = v^{4/(n-2)} dx^i dx^i$. Then for all $\psi \in C^\infty(\Omega_\alpha)$

$$\begin{cases} L_{\hat{g}}(\psi/v) = v^{-(n+2)/(n-2)} L_{g_0}(\psi) & \text{in } \ \Omega_\alpha, \\ B_{\hat{g}}(\psi/v) = v^{-n/(n-2)} B_{g_0}(\psi) & \text{on } \ \partial\Omega_\alpha. \end{cases} \tag{2.29}$$

Let $\psi = v_\alpha$ in (2.29) and write $w_\alpha = v_\alpha / v$, we have

$$\begin{cases} \Delta_{g_0} v_\alpha = v^{(n+2)/(n-2)} (\Delta_{\hat{g}} w_\alpha - a(n) R_{\hat{g}} w_\alpha) & \text{in } \ \Omega_\alpha, \\ \frac{\partial_{g_0} v_\alpha}{\partial \nu} + b(n) H_{g_0} u_\alpha = v^{n/(n-2)} (\frac{\partial_{\hat{g}} w_\alpha}{\partial \nu} + b(n) H_{\hat{g}} w_\alpha) & \text{on } \ \partial\Omega_\alpha. \end{cases} \tag{2.30}$$

Let $\psi = v$ in (2.29), we get

$$\begin{cases} -a(n) R_{\hat{g}} v^{(n+2)/(n-2)} = \Delta_{g_0} v \\ b(n) H_{\hat{g}} v^{n/n-2} = \frac{\partial_{g_0} v}{\partial \nu} + b(n) H_{g_0} v. \end{cases} \tag{2.31}$$

Combining (2.30), (2.31) with (2.22), we have

$$\begin{cases} -\Delta_{\hat{g}} w_\alpha = \xi_\alpha w_\alpha^{p-1} + \Delta_{g_0} v / v^{(n+2)/(n-2)} \cdot w_\alpha & \text{in } \ \Omega_\alpha, \\ \frac{\partial_{\hat{g}} w_\alpha}{\partial \nu} = -Z(\int_{\partial\Omega} v_\alpha^q)^{2/q-1} w_\alpha^{q-1} - (\alpha\mu_\alpha / v^{2/(n-2)} + \frac{\partial v}{\partial \nu} / v^{n/(n-2)}) w_\alpha & \text{on } \ \partial\Omega_\alpha. \end{cases} \tag{2.32}$$

By a similar calculation to the proof of Lemma 2.3.1 in [22] (see also [18]), we have, as α large enough, that

$$\alpha\mu_\alpha / v^{2/(n-2)} + \frac{\partial v}{\partial \nu} / v^{n/(n-2)} \geq 0, \quad x \in \partial\Omega_\alpha.$$

Thus w_α satisfies

$$\begin{cases} -\Delta_{\hat{g}} w_\alpha \leq \xi_\alpha w_\alpha^{p-1} & \text{in } \ \Omega_\alpha, \\ \frac{\partial_{\hat{g}} w_\alpha}{\partial \nu} \leq -Z(\int_{\partial\Omega} v_\alpha^q)^{2/q-1} w_\alpha^{q-1} & \text{on } \ \partial\Omega_\alpha. \end{cases} \tag{2.33}$$

Note Z may be a negative number here.

Define $\Theta_\alpha = \{y : y = x/|x|^2, \ x \in \Omega_\alpha\}, y = x/|x|^2, W_\alpha(y) = w_\alpha(x)$ and $\tilde{g}(y) = \hat{g}(x)$. Then we have

$$\begin{cases} -\Delta_{\tilde{g}} W_\alpha \leq \xi_\alpha W_\alpha^{p-1} & \text{in } \ \Theta_\alpha, \\ \frac{\partial_{\tilde{g}} W_\alpha}{\partial \nu} \leq -Z(\int_{\partial\Omega} v_\alpha^q)^{2/q-1} W_\alpha^{q-1} & \text{on } \ \partial\Theta_\alpha. \end{cases} \tag{2.34}$$

If we write $\tilde{g}(y) = \tilde{g}_{ij}(y) dy^i dy^j$, due to $\tilde{g}(y) = \hat{g}(x) = v^{4/(n-2)} g_0$, we know $\tilde{g}_{ij} = |x|^4 v^{4/(n-2)} \delta_{ij}$. Thus, there exists a $C > 0$, such that $1/C \leq \tilde{g}_{ij}(y) \leq C$ for $y \in \Theta_\alpha \cap B_1(0)$.

Using these notations, we rewrite (2.24) in the following setting.

Lemma 2.4

$$\lim_{\alpha\to\infty} \int_{\Theta_\alpha} |W_\alpha - 1|^p dv_{\tilde{g}} = \lim_{\alpha\to\infty} \int_{\partial\Theta_\alpha} |W_\alpha - 1|^q ds_{\tilde{g}} = 0. \tag{2.35}$$

Now we focus on proving the following proposition.

Proposition 2.2 *There exists a constant $C > 0$ such that, $\forall \alpha > \bar{\alpha}$,*

$$v_\alpha \leq Cv \quad \text{for} \ \ x \in \bar{\Omega}_\alpha, \tag{2.36}$$

where v is given by (2.26) or (2.27), depending on the value of Z.

Proof. We only need to show that (2.36) holds for α large, thus, without loss of generality, we can assume $w_\alpha = v_\alpha/v$ satisfies (2.33). Due to the fact that $v_\alpha \to v$ in $C^3(\overline{\Omega_\alpha} \cap B_R(0))$ for all $R > 1$, we only need to show that (2.36) holds for $|x|$ large, that is to show $W_\alpha \le C$ for $|y|$ small.

Let η be some smooth cutoff function with compact support in $B_1(0)$. Multiplying (2.34) by $W_\alpha^k \eta^2$ for $k > 1$ and integrating by parts, we obtain (since $\xi_\alpha,\ Z(\int_{\partial\Omega_\alpha} v_\alpha^q)^{2/q-1} \le C$)

$$\begin{aligned}
&\int_{\Theta_\alpha} \nabla_{\tilde g} W_\alpha \cdot \nabla_{\tilde g}(W_\alpha^k \eta^2) dv_{\tilde g} \\
&\le \xi_\alpha \int_{\Theta_\alpha} W_\alpha^{p-1+k}\eta^2 dv_{\tilde g} - Z(\int_{\partial\Omega_\alpha} v_\alpha^q)^{2/q-1} \cdot \int_{\partial\Theta_\alpha} W_\alpha^{q-1+k}\eta^2 ds_{\tilde g} \\
&\le C \int_{\Theta_\alpha} W_\alpha^{p-1+k}\eta^2 dv_{\tilde g} + C \int_{\partial\Theta_\alpha} W_\alpha^{q-1+k}\eta^2 ds_{\tilde g}.
\end{aligned}$$

Direct computation yields:

$$\begin{aligned}
&\int_{\Theta_\alpha} \nabla_{\tilde g} W_\alpha \cdot \nabla_{\tilde g}(W_\alpha^k \eta^2) dv_{\tilde g} \\
&= \tfrac{4k}{(k+1)^2} \int_{\Theta_\alpha} |\nabla_{\tilde g}(W_\alpha^{k+1/2}\eta)|^2 dv_{\tilde g} + \tfrac{k-1}{(k+1)^2} \int_{\Theta_\alpha} W_\alpha^{k+1}\Delta_{\tilde g}\eta^2 dv_{\tilde g} \\
&- \tfrac{4k}{(k+1)^2} \int_{\Theta_\alpha} W_\alpha^{k+1}|\nabla_{\tilde g}\eta|^2 dv_{\tilde g} - \tfrac{k-1}{(k+1)^2} \int_{\partial\Theta_\alpha} W_\alpha^{k+1}\nabla_{\tilde g}\eta^2 \cdot \nu ds_{\tilde g}.
\end{aligned}$$

We derive from the last two inequalities that

$$\begin{aligned}
&\int_{\Theta_\alpha} |\nabla_{\tilde g}(W_\alpha^{k+1/2}\eta)|^2 dv_{\tilde g} \\
&\le \int_{\Theta_\alpha} W_\alpha^{k+1}(|\Delta_{\tilde g}\eta^2| + |\nabla_{\tilde g}\eta|^2) dv_{\tilde g} + \tfrac{k-1}{4k} \int_{\partial\Theta_\alpha} W_\alpha^{k+1}\nabla_{\tilde g}\eta^2 \cdot \nu ds_{\tilde g} \\
&+ \tfrac{C(k+1)^2}{4k} \int_{\Theta_\alpha} W_\alpha^{p-1+k}\eta^2 dv_{\tilde g} + \tfrac{C(k+1)^2}{4k} \int_{\partial\Theta_\alpha} W_\alpha^{q-1+k}\eta^2 ds_{\tilde g}.
\end{aligned} \tag{2.37}$$

Set, for $0 < \overline{\delta} < 1/2$ ($\overline{\delta}$ will be chosen later),

$$R_i = (1 + \frac{1}{2^{i-1}})\overline{\delta}, \qquad i = 1, 2, 3, \cdots. \tag{2.38}$$

we can choose some smooth cutoff function η_i satisfying

$$\begin{cases} \eta_i(y) = 1,\ |y| < R_{i+1};\ \eta_i(y) = 0,\ |y| > R_i; \\ |\nabla_{\tilde g}\eta_i| \le C2^i, \quad |\nabla_{\tilde g}^2\eta_i| \le C4^i. \end{cases}$$

Taking $\eta = \eta_i$ in (2.37) and using Sobolev embedding theorem (see Appendix A in [17]) we reach

$$\begin{aligned}
&\left\{ \int_{\Theta_\alpha \cap B_{R_i}} (W_\alpha^{(k+1)/2}\eta_i)^p dv_{\tilde g} \right\}^{2/p} + \left\{ \int_{\partial\Theta_\alpha \cap B_{R_i}} (W_\alpha^{(k+1)/2}\eta_i)^q ds_{\tilde g} \right\}^{2/q} \\
\le\ & 4^i C \int_{\Theta_\alpha \cap B_{R_i}} W_\alpha^{k+1} dv_{\tilde g} + 2^i C \int_{\partial\Theta_\alpha \cap B_{R_i}} W_\alpha^{k+1} ds_{\tilde g} \\
&+ \tfrac{C(k+1)^2}{k} \int_{\Theta_\alpha \cap B_{R_i}} W_\alpha^{p-1+k} dv_{\tilde g} + \tfrac{C(k+1)^2}{k} \int_{\partial\Theta_\alpha \cap B_{R_i}} W_\alpha^{q-1+k} ds_{\tilde g}.
\end{aligned} \tag{2.39}$$

It follows from Lemma 2.4 that for any $\epsilon_0 > 0$, there exists a $\delta_0 > 0$ such that for any $\delta < \delta_0$,

$$\int_{\Theta_\alpha \cap B_\delta(0)} W_\alpha^p dv_{\tilde{g}} + \int_{\partial\Theta_\alpha \cap B_\delta(0)} W_\alpha^q ds_{\tilde{g}} < \epsilon_0.$$

By the standard Moser iteration, we know for any $\bar{s} > p$, there exists $\delta_1 > 0$, such that for any $p \leq s < \bar{s}$, $\delta < \delta_1$,

$$\int_{\Theta_\alpha \cap B_\delta(0)} W_\alpha^s dv_{\tilde{g}} + \int_{\partial\Theta_\alpha \cap B_\delta(0)} W_\alpha^s ds_{\tilde{g}} < C(\bar{s}). \tag{2.40}$$

Choose $s_0 \in (p, \bar{s})$ and s_0 close to p. Let $r_0 = s_0/(p-2)$, $\beta = p(r_0-1)/(2r_0)$ and $\frac{t_0}{t_0-1} = \frac{q}{2\beta}$. We can check $\beta > 1$. Also as s_0 is close to p, β is close to 1. Therefore, we can make $2\beta < q$ and $(q-2)t_0 < \bar{s}$ after we choose a suitable s_0. Choose $2\bar{\delta} < \delta_1$. By Hölder inequality, we know

$$\int_{\Theta_\alpha \cap B_{R_i}} W_\alpha^{p-1+k} dv_{\tilde{g}} \leq \Big(\int_{\Theta_\alpha \cap B_{R_i}} W_\alpha^{(k+1)r_0/(r_0-1)} dv_{\tilde{g}}\Big)^{(r_0-1)/r_0} \Big(\int_{\Theta_\alpha \cap B_{R_i}} W_\alpha^{s_0}\Big)^{\frac{1}{r_0}}$$

and

$$\int_{\partial\Theta_\alpha \cap B_{R_i}} W_\alpha^{q-1+k} ds_{\tilde{g}} \leq \Big(\int_{\partial\Theta_\alpha \cap B_{R_i}} W_\alpha^{(k+1)t_0/(t_0-1)} ds_{\tilde{g}}\Big)^{(t_0-1)/t_0} \Big(\int_{\partial\Theta_\alpha \cap B_{R_i}} W_\alpha^{(q-2)t_0}\Big)^{\frac{1}{t_0}}.$$

Combining the above two inequalities with (2.40) we have

$$\int_{\Theta_\alpha \cap B_{R_i}} W_\alpha^{p-1+k} dv_{\tilde{g}} \leq C\Big(\int_{\Theta_\alpha \cap B_{R_i}} W_\alpha^{(k+1)r_0/(r_0-1)} dv_{\tilde{g}}\Big)^{(r_0-1)/r_0} \tag{2.41}$$

and

$$\int_{\partial\Theta_\alpha \cap B_{R_i}} W_\alpha^{q-1+k} ds_{\tilde{g}} \leq C\Big(\int_{\partial\Theta_\alpha \cap B_{R_i}} W_\alpha^{(k+1)t_0/(t_0-1)} ds_{\tilde{g}}\Big)^{(t_0-1)/t_0}. \tag{2.42}$$

Also, from Hölder inequality,

$$\int_{\partial\Theta_\alpha \cap B_{R_i}} W_\alpha^{1+k} ds_{\tilde{g}} \leq C\Big(\int_{\partial\Theta_\alpha \cap B_{R_i}} W_\alpha^{(k+1)t_0/(t_0-1)} ds_{\tilde{g}}\Big)^{(t_0-1)/t_0}. \tag{2.43}$$

Set $p_0 = 2r_0/(r_0-1) < p$, $p_i = \beta p_{i-1} = \beta^{i-1}p$, $q_i = p_i(r_0-1)/r_0 = 2\beta^i$, for $i \geq 1$. Taking $k = q_i - 1$ (for $i \geq 1$) in (2.39), and using (2.41), (2.42) and (2.43), we obtain

$$\begin{aligned} & \|W_\alpha\|^{q_i}_{p_{i+1},\Theta_\alpha \cap B_{R_{i+1}}} + \|W_\alpha\|^{q_i}_{\bar{q}_{i+1},\partial\Theta_\alpha \cap B_{R_{i+1}}} \\ \leq\ & (4^i C + \tfrac{q_i^2 C}{(q_i-1)})(\|W_\alpha\|^{q_i}_{p_i,\Theta_\alpha \cap B_{R_i}} + \|W_\alpha\|^{q_i}_{\bar{q}_i,\partial\Theta_\alpha \cap B_{R_i}}), \end{aligned}$$

where $\bar{q}_i = q_i \cdot \frac{t_0}{t_0-1}$ for $i \geq 1$. Since $\beta > 1$, we have $(a^\beta + b^\beta)^{2/\beta} \leq a + b$. It follows that

$$\begin{array}{rl} & (\|W_\alpha\|^{q_{i+1}}_{p_{i+1},\Theta_\alpha\cap B_{R_{i+1}}} + \|W_\alpha\|^{q_{i+1}}_{\bar{q}_{i+1},\partial\Theta_\alpha\cap B_{R_{i+1}}})^{1/q_{i+1}} \\ \leq & (4^i C + \frac{q_i^2 C}{q_i-1})^{1/q_i}(\|W_\alpha\|^{q_i}_{p_i,\Theta_\alpha\cap B_{R_i}} + \|W_\alpha\|^{q_i}_{\bar{q}_i,\partial\Theta_\alpha\cap B_{R_i}})^{1/q_i}. \end{array} \tag{2.44}$$

As in [17], one can easily check that

$$\prod_{i=1}^{\infty}(4^i C + \frac{q_i^2 C}{q_i - 1})^{1/q_i} \leq C < \infty,$$

thus

$$\begin{array}{rl} \|W_\alpha\|_{p_{i+1},\Theta_\alpha\cap B_{R_{i+1}}} & \leq C(\|W_\alpha\|^{2\beta}_{p_1,\Theta_\alpha\cap B_{R_1}} + \|W_\alpha\|^{2\beta}_{q_1,\partial\Theta_\alpha\cap B_{R_1}})^{1/(2\beta)} \\ & = C(\|W_\alpha\|^{2\beta}_{p,\Theta_\alpha\cap B_{R_1}} + \|W_\alpha\|^{2\beta}_{q,\partial\Theta_\alpha\cap B_{R_1}})^{1/(2\beta)} \\ & \leq C_1. \end{array}$$

Sending i to ∞, we have

$$\|W_\alpha\|_{L^\infty(\Theta_\alpha\cap B_{\bar{\delta}})} \leq C(\bar{\delta}). \tag{2.45}$$

Therefore we complete the proof of Proposition 2.2.

Let $Q_\alpha \in \partial\Omega$ be the closest point to x_α. By choosing an appropriate coordinate system centered at Q_α, we can assume without loss of generality that $Q_\alpha = 0$, $g_{ij}(0) = \delta_{ij}$, $B_1^+(0) \subset \Omega$, $\{(x', 0) \ : \ |x'| < 1\} \subset \partial\Omega$.

Let $R_\alpha = 1/(\alpha\mu_\alpha)$, $h_\alpha = g_{ij}(\mu_\alpha x)dx^i dx^j$ in $B^+_{10R_\alpha}(0)$, and

$$\bar{v}_\alpha(x) = \mu_\alpha^{(n-2)/2} u_\alpha(\mu_\alpha x + x_\alpha), \qquad \text{for} \quad x \in B^+_{10R_\alpha}(0).$$

It follows from (2.23) and (2.22) that $R_\alpha \to \infty$ as $\alpha \to \infty$, and $\bar{v}_\alpha$ satisfies

$$\left\{ \begin{array}{ll} -\Delta_{h_\alpha}\bar{v}_\alpha = \xi_\alpha \bar{v}_\alpha^{p-1} & \text{in} \ \ B^+_{10R_\alpha}(0) \\ \frac{\partial_{h_\alpha}\bar{v}_\alpha}{\partial\nu} = -Z(\int_{\partial\Omega} u_\alpha^q)^{2/q-1}\bar{v}_\alpha^{q-1} - \alpha\mu_\alpha\bar{v}_\alpha & \text{on} \ \ \{(x', 0) \ : \ |x'| < 10R_\alpha\} \\ 0 < \bar{v}_\alpha \leq \mu_\alpha^{(n-2)/2} u_\alpha(0). & \end{array} \right. \tag{2.46}$$

Clearly,

$$|h_\alpha^{ij}(x) - \delta^{ij}| \leq C|\mu_\alpha x|, \quad |\Gamma^k_{ij}(x)| \leq C\mu_\alpha \ \text{ in } \ B^+_{10R_\alpha}(0), \tag{2.47}$$

where Γ^k_{ij} is the Christoffel symbol of h_α.

As being explained before,

$$\lim_{\alpha\to\infty} \|\bar{v}_\alpha - v_T\|_{C^2(\overline{B_R^+(0)})} = 0, \quad \forall R > 1, \tag{2.48}$$

where $v_T = v(x', x_n - T)$, v is given in (2.26) or (2.27). It is not difficult to see from Proposition 2.2 that as $\alpha > \bar{\alpha}$,

$$\bar{v}_\alpha(x) \leq \frac{C}{1 + |x|^{n-2}} \quad \text{for} \quad x \in \overline{B^+_{10R_\alpha}}(0). \tag{2.49}$$

Note $Z(\int_{\partial\Omega} u_\alpha^q)^{2/q-1} \leq C$, we can show, exactly in the same way as in [17], the following estimates on the first and second derivative of $\bar{v}_\alpha$.

Proposition 2.3 *For all $\alpha \geq \bar{\alpha}$, $x \in \overline{B^+_{R_\alpha}(0)}$, we have*

$$|\nabla \bar{v}_\alpha(x)| \leq \frac{C}{1 + |x|^{n-1}}, \quad |\nabla^2 \bar{v}_\alpha(x)| \leq \frac{C}{1 + |x|^n},$$

where $|\nabla^2 \bar{v}_\alpha| = \sum_{i,j=1}^n |\partial^2 \bar{v}_\alpha / \partial x^i \partial x^j|$, and C is some constant independent of α and x.

For $n = 3$, we need to obtain an appropriate lower bound of $\bar{v}_\alpha$.

Proposition 2.4 *For $n = 3$, as α large enough,*

$$\bar{v}_\alpha(x) \geq \frac{1}{C(1 + |x|)}, \quad \forall\ x \in \overline{B^+_{R_\alpha^{1/4}}(0)},$$

where $C > 0$ is some constant independent of α.

Proof. If $Z \leq 0$, Proposition 2.4 can be provn exactly in the same way as that in [17], therefore we will focus on the case of $Z > 0$ here. The proof is slight different from that in [17]. Due to $Z > 0$, we know $T > 0$. Without loss of generality, we can assume $T \geq 1$. In this case, we need to use more accurate boundary condition in (2.46). In the following, α is always assumed to be suitable large.

Let $\bar{x} = (0, ..., 0, 1)$ and

$$G_\alpha(x) = \frac{1}{|x - \bar{x}|} - \frac{1}{R_\alpha^{1/2} |x - \bar{x}|^{1/2}} \quad \text{in} \quad B_{R_\alpha^{1/3}}(\bar{x}) \setminus B_2(\bar{x}).$$

It is easy to see that

$$\frac{1}{2|x - \bar{x}|} \leq G_\alpha(x) \leq \frac{1}{|x - \bar{x}|} \quad \text{in} \quad B_{R_\alpha^{1/3}}(\bar{x}) \setminus B_2(\bar{x}).$$

As in [17], by using (2.47), one can check that $\Delta_{h_\alpha} G_\alpha \geq 0$ for $x \in B_{R_\alpha^{1/3}}(\bar{x}) \setminus B_{2T}(\bar{x})$.

Also, from (2.47), we know that for all $x=(x',0),\quad 1<|x'|<R_\alpha^{1/3}$, there exists a constant $C_1>0$ such that

$$\frac{\partial_{h_\alpha}}{\partial\nu}(G_\alpha)\le-\frac{1}{C_1}\cdot\frac{1}{|x-\bar{x}|^3}\le-\frac{1}{C_1}\cdot G_\alpha^3.$$

We will use the maximum principle and Hopf lemma on $A=\{x\in\mathbb{R}^n_+\ :\ 2T<|x-\bar{x}|<R_\alpha^{1/3}\}$. Let $\Sigma_1=\partial A\cap\{x_n=0\}$, $\Sigma_2=\partial A\cap\{|x-\bar{x}|=2T\}$, and $\Sigma_3=\partial A\cap\{|x-\bar{x}|=R_\alpha^{1/3}\}$. Choose $0<\tau_1<1$ small enough such that $\tau_1G_\alpha\le\bar{v}_\alpha$ on Σ_2. Note $Z(\int_{\partial\Omega}u_\alpha^q)^{2/q-1}\le C$ (in the remains of the proof of Proposition 2.4, we always take C as the same positive constant), we choose $\tau_2<\tau_1$ small enough such that $1/(C_1\tau_2^2)\ge C$. Let $H_\alpha=\tau_2G_\alpha-\max_{\Sigma_3}(\tau_2G_\alpha)$. One can check that $\bar{v}_\alpha-H_\alpha$ satisfies

$$\begin{cases}\Delta_{h_\alpha}(\bar{v}_\alpha-H_\alpha)\le 0 & \text{in }\ A,\\ \bar{v}_\alpha-H_\alpha\ge 0 & \text{on }\ \ \Sigma_2\cup\Sigma_3,\\ \frac{\partial_{h_\alpha}(\bar{v}_\alpha-H_\alpha)}{\partial\nu}>C(H_\alpha^3-\bar{v}_\alpha^3) & \text{on }\ \Sigma_1.\end{cases}$$

It follows from the maximum principle and Hopf lemma that

$$\bar{v}_\alpha\ge H_\alpha\quad\text{in }\ A.$$

Consequently, for all $x\in B^+_{R_\alpha^{1/4}}(0)\setminus B^+_{2T}(\bar{x})$,

$$\bar{v}_\alpha(x)\ge H_\alpha(x)\ge\frac{\tau_2}{2|x-\bar{x}|}-\frac{\tau_2}{R_\alpha^{1/3}}\ge\frac{\tau_2}{4|x-\bar{x}|}.$$

For $|x-\bar{x}|\le 2T$, Proposition 2.4 follows from (2.48).

Let $B^+_{R_\alpha}=B_{R_\alpha}(0)\cap\mathbb{R}^n_+$, $\Gamma_1=\partial B^+_{R_\alpha}\cap\partial\mathbb{R}^n_+$, $\Gamma_2=\partial B^+_{R_\alpha}\cap\mathbb{R}^n_+$. We always use dV for the volume element of the standard Euclidean metric, dS for the surface element of the standard Euclidean metric, ν for the unit outer normal vector of the corresponding surface with respect to the specified metrics, and "$\cdot$" for the inner product under the standard Euclidean metric. As in [17], we have the following identity.

$$\int_{B^+_{R_\alpha}}\Delta\bar{v}_\alpha(\nabla\bar{v}_\alpha\cdot x)dV+\frac{n-2}{2}\int_{B^+_{R_\alpha}}\bar{v}_\alpha\Delta\bar{v}_\alpha dV=J(R_\alpha,\bar{v}_\alpha)+I(R_\alpha,\bar{v}_\alpha),\tag{2.50}$$

where

$$J(R_\alpha,\bar{v}_\alpha)=\frac{1}{2}\int_{\Gamma_2}\{|\frac{\partial\bar{v}_\alpha}{\partial\nu}|^2|x|-|\partial_{tan}\bar{v}_\alpha|^2|x|+(n-2)\frac{\partial\bar{v}_\alpha}{\partial\nu}\bar{v}_\alpha\}dS,\tag{2.51}$$

$$I(R_\alpha,\bar v_\alpha)=\frac{1}{2}\int_{\Gamma_1}\{2(\sum_{i=1}^{n-1}x_i\frac{\partial\bar v_\alpha}{\partial x_i})\frac{\partial\bar v_\alpha}{\partial\nu}+(n-2)\frac{\partial\bar v_\alpha}{\partial\nu}\bar v_\alpha\}dS. \tag{2.52}$$

Replacing $\Delta\bar v_\alpha$ in (2.50) by

$$\Delta\bar v_\alpha=\Delta_{h_\alpha}\bar v_\alpha-(h_\alpha^{ij}-\delta^{ij})\partial_{ij}\bar v_\alpha+h_\alpha^{ij}\Gamma_{ij}^k\partial_k\bar v_\alpha,$$

we have

$$\begin{array}{l}
-\int_{B^+_{R_\alpha}}(x^i\partial_i\bar v_\alpha)\Delta_{h_\alpha}\bar v_\alpha dV-\frac{n-2}{2}\int_{B^+_{R_\alpha}}\bar v_\alpha\Delta_{h_\alpha}\bar v_\alpha dV\\
+\int_{B^+_{R_\alpha}}(x^k\partial_k\bar v_\alpha)(h_\alpha^{ij}-\delta^{ij})\partial_{ij}\bar v_\alpha dV-\int_{B^+_{R_\alpha}}(x^l\partial_l\bar v_\alpha)(h_\alpha^{ij}\Gamma_{ij}^k\partial_k\bar v_\alpha)dV\\
+\frac{n-2}{2}\int_{B^+_{R_\alpha}}\bar v_\alpha(h_\alpha^{ij}-\delta^{ij})\partial_{ij}\bar v_\alpha dV-\frac{n-2}{2}\int_{B^+_{R_\alpha}}\bar v_\alpha(h_\alpha^{ij}\Gamma_{ij}^k)\partial_k\bar v_\alpha dV\\
=-J(R_\alpha,\bar v_\alpha)-I(R_\alpha,\bar v_\alpha).
\end{array}$$

Using equation (2.46), we get

$$A(h_\alpha,\bar v_\alpha)=-J(R_\alpha,\bar v_\alpha)-I(R_\alpha,\bar v_\alpha), \tag{2.53}$$

where

$$\begin{array}{rl}
A(h_\alpha,\bar v_\alpha)= & \frac{\xi_\alpha}{p}\int_{\Gamma_2}\bar v_\alpha^p|x|dS\\
& +\int_{B^+_{R_\alpha}}(x^k\partial_k\bar v_\alpha)(h_\alpha^{ij}-\delta^{ij})\partial_{ij}\bar v_\alpha dV-\int_{B^+_{R_\alpha}}(x^l\partial_l\bar v_\alpha)(h_\alpha^{ij}\Gamma_{ij}^k\partial_k\bar v_\alpha)dV\\
& +\frac{n-2}{2}\int_{B^+_{R_\alpha}}\bar v_\alpha(h_\alpha^{ij}-\delta^{ij})\partial_{ij}\bar v_\alpha dV-\frac{n-2}{2}\int_{B^+_{R_\alpha}}\bar v_\alpha(h_\alpha^{ij}\Gamma_{ij}^k)\partial_k\bar v_\alpha dV.
\end{array}$$

By (2.47), we know

$$\begin{array}{rl}
A(h_\alpha,\bar v_\alpha)= & O(\int_{\Gamma_2}\bar v_\alpha^p|x|dS)\\
& +O(\int_{B^+_{R_\alpha}}\mu_\alpha|x|^2|\nabla\bar v_\alpha|\ |\nabla^2\bar v_\alpha|dV)+O(\int_{B^+_{R_\alpha}}\mu_\alpha|x||\nabla\bar v_\alpha|^2dV)\\
& +O(\int_{B^+_{R_\alpha}}\mu_\alpha|x|\bar v_\alpha|\nabla^2\bar v_\alpha|dV)+O(\int_{B^+_{R_\alpha}}\mu_\alpha\bar v_\alpha|\nabla\bar v_\alpha|dV)
\end{array} \tag{2.54}$$

We simplify $I(R_\alpha,\bar v_\alpha)$ by using equation (2.46). It is easy to see from (2.47) that

$$\frac{\partial_{h_\alpha}\bar v_\alpha}{\partial\nu}=\frac{\partial\bar v_\alpha}{\partial\nu}+O(\mu_\alpha|x'|\ |\nabla\bar v_\alpha|),\qquad \text{on }\Gamma_1.$$

It follows that

$$\begin{array}{rl}
2I(R_\alpha,\bar v_\alpha)= & \int_{\Gamma_1}\{2(\sum_{i=1}^{n-1}x_i\frac{\partial\bar v_\alpha}{\partial x_i})\frac{\partial_{h_\alpha}\bar v_\alpha}{\partial\nu}+(n-2)\frac{\partial_{h_\alpha}\bar v_\alpha}{\partial\nu}\bar v_\alpha\}dS\\
& +O(\int_{\Gamma_1}[\mu_\alpha|x'|^2\ |\nabla\bar v_\alpha|^2+\mu_\alpha|x'|\bar v_\alpha|\nabla\bar v_\alpha|]dS).
\end{array} \tag{2.55}$$

Using the boundary condition in (2.46), we have

$$\begin{aligned}
&\int_{\Gamma_1}\{2(\sum_{i=1}^{n-1} x_i\frac{\partial \bar v_\alpha}{\partial x_i})\frac{\partial_{h_\alpha}\bar v_\alpha}{\partial\nu}+(n-2)\frac{\partial_{h_\alpha}\bar v_\alpha}{\partial\nu}\bar v_\alpha\}dS\\
&=\int_{\Gamma_1}\{-2(\sum_{i=1}^{n-1} x_i\frac{\partial \bar v_\alpha}{\partial x_i})(\alpha\mu_\alpha\bar v_\alpha+Z(\int_{\partial\Omega}u_\alpha^q)^{2/q-1}\bar v_\alpha^{q-1})\\
&-(n-2)\alpha\mu_\alpha\bar v_\alpha^2-(n-2)Z(\int_{\partial\Omega}u_\alpha^q)^{2/q-1}\bar v_\alpha^q\}dS\\
=\ &\alpha\mu_\alpha\int_{\Gamma_1}\bar v_\alpha^2dS-\int_{\partial\Gamma_1}\alpha\mu_\alpha\bar v_\alpha^2|x|dS-\frac{2Z}{q}\cdot(\int_{\partial\Omega}u_\alpha^q)^{2/q-1}\int_{\partial\Gamma_1}\bar v_\alpha^q|x|dS.
\end{aligned}$$

Thus

$$\begin{aligned}
I(R_\alpha,\bar v_\alpha)\ &=\frac{\alpha\mu_\alpha}{2}\int_{\Gamma_1}\bar v_\alpha^2dS+O(\int_{\partial\Gamma_1}(\alpha\mu_\alpha\bar v_\alpha^2|x|+\bar v_\alpha^q|x|)dS)\\
&+O(\int_{\Gamma_1}[\mu_\alpha|x'|^2|\nabla\bar v_\alpha|^2+\mu_\alpha|x'|\bar v_\alpha\ |\nabla\bar v_\alpha|]dS).
\end{aligned}\tag{2.56}$$

Clearly,

$$J(R_\alpha,\bar v_\alpha)=O(\int_{\Gamma_2}(|x||\nabla\bar v_\alpha|^2+\bar v_\alpha\ |\nabla\bar v_\alpha|)dS).\tag{2.57}$$

We can rewrite (2.53) as the following Pohozaev type identity:

$$\begin{aligned}
\alpha\mu_\alpha\int_{\Gamma_1}\bar v_\alpha^2dS=\ &O(\int_{\Gamma_2}\bar v_\alpha^p|x|dS)\\
&+O(\int_{B^+_{R_\alpha}}\mu_\alpha|x|^2|\nabla\bar v_\alpha|\ |\nabla^2\bar v_\alpha|dV)+O(\int_{B^+_{R_\alpha}}\mu_\alpha|x||\nabla\bar v_\alpha|^2dV)\\
&+O(\int_{B^+_{R_\alpha}}\mu_\alpha|x|\bar v_\alpha|\nabla^2\bar v_\alpha|dV)+O(\int_{B^+_{R_\alpha}}\mu_\alpha\bar v_\alpha|\nabla\bar v_\alpha|dV)\\
&+O(\int_{\Gamma_2}(|x||\nabla\bar v_\alpha|^2+\bar v_\alpha\ |\nabla\bar v_\alpha|)dS)\\
&+O(\int_{\partial\Gamma_1}(\alpha\mu_\alpha\bar v_\alpha^2|x|+\bar v_\alpha^q|x|)dS)\\
&+O(\int_{\Gamma_1}[\mu_\alpha|x'|^2|\nabla\bar v_\alpha|^2+\mu_\alpha|x'|\bar v_\alpha\ |\nabla\bar v_\alpha|]dS).
\end{aligned}\tag{2.58}$$

We will derive a contradiction from (2.58) by showing that the left hand side is much larger than the right hand side as α tends to infinity.

Similarly as in [17], by using (2.48) and Proposition 2.4, we have

Lemma 2.5 *For $n\geq 3$, there exists some constant $C>0$ independent of α, such that $\int_{\Gamma_1}\bar v_\alpha^2dS>1/C$ for all $\alpha\geq 1$. Moreover for $n=3$, $\int_{\Gamma_1}\bar v_\alpha^2dS\geq(\log R_\alpha)/C$ for all $\alpha\geq 1$.*

Also, by using (2.49), Proposition 2.3 and some elementary calculations, we have

Lemma 2.6 *The following estimates hold.*

$$\int_{\partial\Gamma_1}(\alpha\mu_\alpha\bar v_\alpha^2|x|+\bar v_\alpha^q|x|)dS\leq\alpha\mu_\alpha R_\alpha^{3-n},$$

$$\int_{\Gamma_1}(\mu_\alpha|x'|^2|\nabla\bar v_\alpha|^2+\mu_\alpha|x'|\bar v_\alpha\ |\nabla\bar v_\alpha|)dS\leq\left\{\begin{array}{rl} C\mu_\alpha\log R_\alpha, & n=3,\\ C\mu_\alpha, & n\geq 4,\end{array}\right.$$

$$\int_{\Gamma_2}(|x||\nabla\bar{v}_\alpha|^2+\bar{v}_\alpha\ |\nabla\bar{v}_\alpha|)dS\le C(\alpha\mu_\alpha)^{n-2},$$

$$\int_{\Gamma_2}\bar{v}_\alpha^p|x|dS\le C(\alpha\mu_\alpha)^n,$$

$$\int_{B^+_{R_\alpha}}(\mu_\alpha|x|^2|\nabla\bar{v}_\alpha|\ |\nabla^2\bar{v}_\alpha|+\mu_\alpha|x||\nabla\bar{v}_\alpha|^2)dV\le\begin{cases} C\mu_\alpha\log R_\alpha, & n=3,\\ C\mu_\alpha, & n\ge 4,\end{cases}$$

$$\int_{B^+_{R_\alpha}}(\mu_\alpha|x|\bar{v}_\alpha|\nabla^2\bar{v}_\alpha|+\mu_\alpha\bar{v}_\alpha|\nabla\bar{v}_\alpha|)dV\le\begin{cases} C\mu_\alpha\log R_\alpha, & n=3,\\ C\mu_\alpha, & n\ge 4.\end{cases}$$

Proof of Theorem 0.2. From Lemma 2.5 and Lemma 2.6 we know that the left hand side is clearly much larger than the right hand side in (2.58) as α tends to infinity. Therefore we derive a contradiction basing on the assumption (2.17).

3 Compact manifold with boundary

Let (M,g) be a compact Riemannian manifold with smooth boundary ∂M and dimension $n\ge 3$. In this section we sketch the proof of Theorem 0.3.

First we show a rough inequality as in Section 2.

Proposition 3.1 *Let $Z\in(-1/S,Z_0]$. For any $\delta>0$, there exists $D(\delta)>0$ such that $\forall u\in H^1(M)$,*

$$\begin{aligned}&(\textstyle\int_M|u|^p dv_g)^{\frac{2}{p}}\\ &\le(S_1(Z)+\delta)\Big(\textstyle\int_M|\nabla u|^2dv_g+Z(\int_{\partial M}|u|^q ds_g)^{\frac{2}{q}}\Big)+D(\delta)\Big(\int_{\partial M}u^2ds_g+\int_M u^2dv_g\Big).\end{aligned}\tag{3.1}$$

Proof. We prove this proposition by contradiction. The proof is quite similar to that of Proposition 2.1, we sketch it below.

Assume that (3.1) is not true, that is, there exists some $\delta_2>0$ such that $\forall\alpha>1$,

$$\begin{aligned}\inf_{u\in H^1(M)\backslash\{0\}}I_{Z,\alpha}(u)\ &:=\inf_{u\in H^1(M)\backslash\{0\}}\frac{\int_M|\nabla_g u|^2+Z(\int_{\partial M}|u|^q)^{2/q}+\alpha\int_{\partial M}u^2+\alpha\int_M u^2}{(\int_M|u|^p)^{2/p}}\\ &:=\xi_{Z,\alpha}<\tfrac{1}{S_1(Z)+\delta_2}.\end{aligned}\tag{3.2}$$

Lemma 3.1 *$\forall \delta > 0$, there exists a constant $C(M, \delta)$, such that $\forall f \in H^1(M)$, as ϵ small enough,*

$$||f||^2_{p_\epsilon, M} \leq (S_1 + \delta)||\nabla f||^2_{2,M} + C(M, \delta)\Big(||f||^2_{2,M} + ||f||^2_{q_\epsilon, \partial M}\Big). \tag{3.3}$$

Proof. This can be provn from Corollary 1.2 through the partition of unit. We omit the details here.

Lemma 3.2 *As α large enough, if $\inf I_{Z,\alpha}$ satisfies condition (3.2), $\inf I_{Z,\alpha}$ is attained.*

Proof. Due to [17], we know that there exists $\alpha_1 < \infty$ such that

$$||u||^2_{q,\partial M} \leq S||\nabla_g u||^2_{2,M} + \alpha_1 ||u||^2_{2,\partial M}, \quad \forall u \in H^1(M). \tag{3.4}$$

For $u \neq 0$, we define

$$I_\epsilon(u) = \frac{\int_M |\nabla_g u|^2 + Z(\int_{\partial M} |u|^{q_\epsilon})^{2/q_\epsilon} + \alpha \int_{\partial M} u^2 + \alpha \int_M u^2}{(\int_M |u|^{p_\epsilon})^{2/p_\epsilon}}.$$

If $\alpha > \alpha_1$, we know as before that $I_\epsilon(u) \geq 0$. The standard variational method shows that $\exists\ u_\epsilon \geq 0$ with $||u_\epsilon||_{p_\epsilon} = 1$ such that

$$I_\epsilon(u_\epsilon) = \inf I_\epsilon(u) := \xi_\epsilon.$$

Easy to see that u_ϵ satisfies

$$\begin{cases} -\Delta_g u_\epsilon &= \xi_\epsilon u_\epsilon^{p_\epsilon - 1} - \alpha u_\epsilon & \text{in } M \\ \frac{\partial_g u_\epsilon}{\partial \nu} &= -Z(\int_{\partial M} u_\epsilon^{q_\epsilon})^{2/q_\epsilon - 1} u_\epsilon^{q_\epsilon - 1} - \alpha u_\epsilon & \text{on } \partial M. \end{cases} \tag{3.5}$$

Due to [14], we know that there exists $\alpha_2 < \infty$ such that

$$||u||^2_{p,M} \leq S_1 ||\nabla_g u||^2_{2,M} + \alpha_2 ||u||^2_{2,M}, \quad \forall u \in H^1_0(M). \tag{3.6}$$

Set $\alpha_0 = \max\{\alpha_1, \alpha_2\}$. We only need to show that as $\alpha > \alpha_0$, $||u_\epsilon||_\infty \leq C$. Suppose $||u_\epsilon||_\infty \to \infty$ up to a subsequence, by [10], we know that there exists a $x_\epsilon \in M$ such that $u_\epsilon(x_\epsilon) = ||u||_\infty \to \infty$. Set $\mu_\epsilon = (u_\epsilon(x_\epsilon))^{-\frac{p_\epsilon - 2}{2}}$.

Let $(y^1, \cdots, y^{n-1}, y^n)$ denote some geodesic normal coordinates given by the exponential map $\exp_{x_\epsilon}$. In this coordinate system, the metric g is given by $g_{ij}(y)dy^i dy^j$.

For suitably small $\delta_3 > 0$ (independent of ϵ), we define v_ϵ in a neighborhood of $z = 0$ by

$$v_\epsilon(z) = u_\epsilon^{-1}(x_\epsilon)u_\epsilon(exp_{x_\epsilon}(\mu_\epsilon z)), \qquad z \in O_\epsilon \subset \mathbb{R}^n,$$

where

$$O_\epsilon = \{z \in \mathbb{R}^n \ : \ |z| < \delta_3/\mu_\epsilon, \ \ exp_{x_\epsilon}(\mu_\epsilon z) \in M\}. \tag{3.7}$$

We write $\partial O_\epsilon = \Gamma_\epsilon^1 \cup \Gamma_\epsilon^2$, where

$$\Gamma_\epsilon^1 = \{z \in \partial O_\epsilon \ : \ exp_{x_\epsilon}(\mu_\epsilon z) \in \partial M\}, \qquad \Gamma_\epsilon^2 = \{z \in \partial O_\epsilon \ : \ exp_{x_\epsilon}(\mu_\epsilon z) \in M\}.$$

Then v_ϵ satisfies

$$\begin{cases} -\Delta_{g_\epsilon} v_\epsilon = \xi_\epsilon v_\epsilon^{p-1} - \alpha\mu_\epsilon^2 v_\epsilon, & \text{in } O_\epsilon, \\ \frac{\partial_{g_\epsilon} v_\epsilon}{\partial \nu} = -Z(\int_{\partial M} u_\epsilon^{q_\epsilon})^{2/q_\epsilon - 1} v_\epsilon^{q_\epsilon - 1} - \alpha\mu_\epsilon v_\epsilon, & \text{on } \Gamma_\epsilon^1, \\ v_\epsilon(0) = 1, \qquad 0 \le v_\epsilon \le 1, \end{cases} \tag{3.8}$$

where g_ϵ denotes the metric on O_ϵ given by $g_\epsilon = g_{ij}(\mu_\epsilon z)dz^i dz^j$. As in the proof of Lemma 1.1 (here we need to use (3.4)), we can show that as $\alpha > \alpha_0$,

$$0 \le \overline{\lim_{\epsilon\to 0}} \xi_\epsilon := \xi_0 \le \xi_{Z,\alpha} < \frac{1}{S_1(Z) + \delta_2}. \tag{3.9}$$

We claim:

$$\int_{\partial M} u_\epsilon^{q_\epsilon} \ge C > 0. \tag{3.10}$$

If $\int_{\partial M} u_\epsilon^{q_\epsilon} \to 0$ up to a subsequence, using Lemma 3.1, as in the proof of Lemma 1.3, we know $||u_\epsilon - u_0||_{p_\epsilon, M} \to 0$ for some $u_0 \in H_0^1(M)$. It follows that

$$\frac{\int_M |\nabla_g u_0|^2 dv_g + \alpha \int_M u_0^2 dv_g}{(\int_M |u_0|^p dv_g)^{2/p}} < \frac{1}{S_1},$$

this contradicts (3.6) as $\alpha > \alpha_2$.

Also as in the proof of Lemma 1.4, we can show that

$$T := \overline{\lim_{\epsilon\to\infty}} \frac{dist(x_\epsilon, \partial M)}{\mu_\epsilon} < \infty. \tag{3.11}$$

Then we follow the proof of Lemma 2.1 closely and can derive a contradiction to Theorem 0.1. We thereby establish Lemma 3.2.

Due to Lemma 3.2, without loss of generality, we can assume that as $\alpha > \alpha_0$, $\inf I_{Z,\alpha}(u) = I_{Z,\alpha}(u_\alpha)$ with $u_\alpha \ge 0$ and $||u_\alpha||_{p,M} = 1$. Then, we follow the proof of Proposition 2.1 closely and can complete the proof of Proposition 3.1. The only difference is to show $\int_{\partial M} u_\alpha^q \geq C > 0$. But this can be handled similarly to (3.10). We leave these details to interesting readers.

From now on, we begin to prove Theorem 0.3 through an argument by contradiction. Note that we assume $Z < Z_0$, thus $1/S_1(Z) < 1/S_1$.

Suppose that Theorem 0.3 is false, then $\forall \alpha > 1$,

$$\begin{aligned} \inf_{H^1(M)\setminus\{0\}} I_\alpha(u) &= \inf_{H^1(M)\setminus\{0\}} \frac{\int_M |\nabla_g u|^2 + Z(\int_{\partial M} |u|^q)^{2/q} + \alpha \int_{\partial M} u^2 + \alpha \int_M u^2}{(\int_M |u|^p)^{2/p}} \\ &:= \xi_\alpha < \frac{1}{S_1(Z)}. \end{aligned} \tag{3.12}$$

From the proof of Lemma 3.2, we know that as $\alpha > \alpha_0$, under (3.12), $\inf I_\alpha(u)$ is attained. Without loss of generality, we can always assume α suitable large and $\inf I_\alpha(u) = I_\alpha(u_\alpha)$ with $u_\alpha \geq 0$ and $||u_\alpha||_{p,M} = 1$. It is easy to see that u_α satisfies

$$\begin{cases} -\Delta_g u_\alpha &= \xi_\alpha u_\alpha^{p-1} - \alpha u_\alpha \quad\quad \text{in } \ M \\ \frac{\partial_g u_\alpha}{\partial \nu} &= -Z(\int_{\partial M} u_\alpha^q)^{2/q-1} u_\alpha^{q-1} - \alpha u_\alpha, \quad \text{on } \ \partial M. \end{cases} \tag{3.13}$$

Using Proposition 3.1, we have

Lemma 3.3 *As $\alpha \to \infty$,*

$$\alpha ||u_\alpha||^2_{2,\partial M} \to 0, \quad \xi_\alpha \to \frac{1}{S_1(Z)}. \tag{3.14}$$

Because of $1/S_1(Z) < 1/S_1$, as in the proof of (3.10), we have the following.

Lemma 3.4 *There exists a constant $C > 0$ such that*

$$\int_{\partial M} u_\alpha^q \geq C. \tag{3.15}$$

Since u_α satisfies (3.13), due to Cherrier, we know that u_α is smooth up to boundary. Let $u_\alpha(x_\alpha) = ||u||_\infty$ for some $x_\alpha \in \overline{M}$. Set $\mu_\alpha = (u_\alpha(x_\alpha))^{-\frac{p-2}{2}}$. As before, from (3.14) and (3.15) we can show that

$$\alpha \mu_\alpha \to 0, \quad \text{as} \quad \alpha \to \infty. \tag{3.16}$$

Let $(y^1, \cdots, y^{n-1}, y^n)$ denote some geodesic normal coordinates given by the exponential map $\exp_{x_\alpha}$. In this coordinate system, the metric g is given by $g_{ij}(y)dy^i dy^j$.

For a suitable small $\delta_4 > 0$ (independent of α), we define v_α in a neighborhood of $z = 0$ by

$$v_\alpha(z) = u_\alpha^{-1}(x_\alpha)u_\alpha(exp_{x_\alpha}(\mu_\alpha z)), \qquad z \in O_\alpha \subset \mathbb{R}^n,$$

where

$$O_\alpha = \{z \in \mathbb{R}^n \ : \ |z| < \delta_4/\mu_\alpha, \ \ exp_{x_\alpha}(\mu_\alpha z) \in M\}. \tag{3.17}$$

We write $\partial O_\alpha = \Gamma_\alpha^1 \cup \Gamma_\alpha^2$, where

$$\Gamma_\alpha^1 = \{z \in \partial O_\alpha \ : \ exp_{x_\alpha}(\mu_\alpha z) \in \partial M\}, \quad \Gamma_\alpha^2 = \{z \in \partial O_\alpha \ : \ exp_{x_\alpha}(\mu_\alpha z) \in M\}.$$

Then v_α satisfies

$$\begin{cases} -\Delta_{g_\alpha} v_\alpha = \xi_\alpha v_\alpha^{p-1} - \alpha\mu_\alpha^2 v_\alpha, & \text{in } O_\alpha, \\ \frac{\partial_{g_\alpha} v_\alpha}{\partial\nu} = -Z(\int_{\partial M} u_\alpha^{q_\alpha})^{2/q_\alpha - 1} v_\alpha^{q_\alpha - 1} - \alpha\mu_\alpha v_\alpha, & \text{on } \Gamma_\alpha^1, \\ v_\alpha(0) = 1, \qquad 0 \le v_\alpha \le 1, \end{cases} \tag{3.18}$$

where g_α denotes the metric on O_α given by $g_\alpha = g_{ij}(\mu_\alpha z)dz^i dz^j$.

Also let $\overline{\lim}_{\alpha\to\infty} dist(x_\alpha, \partial M)/\mu_\alpha = T$, due to $1/S_1(Z) < 1/S_1$, as in Section 1, we know $T < \infty$.

By standard elliptic estimates, we know $v_\alpha \to v$ in $C^3(B_R(0) \cap \overline{O}_\alpha)$, where $v(x)$ is given by (2.26) or (2.27) (depending on $T > 0$ or $T = 0$). Consequently, as before, we have the following lemma.

Lemma 3.5

$$\lim_{\alpha\to\infty}\int_{O_\alpha} |v_\alpha - v|^p = \lim_{\alpha\to\infty}\int_{O_\alpha} |\nabla_{g_\alpha} v_\alpha - \nabla_{g_\alpha} v|^2 = \lim_{\alpha\to\infty}\int_{\partial O_\alpha} |v_\alpha - v|^q = 0. \tag{3.19}$$

As in [18], by using Lemma 3.5, we have

Proposition 3.2 *There exists a constant $C > 0$ such that,* $\forall \alpha > 1$

$$v_\alpha \le Cv, \quad x \in \bar{M}_\alpha. \tag{3.20}$$

Then following the proof of Theorem 0.2 closely, by using Pohozaev identity, we derive a contradiction, thus complete the proof of Theorem 0.3. We refer [18] and [17] to interesting readers for more details.

4 Some further remarks

In this section, we give some details concerning Remark 2.1 and point out the obstacle by using the current method to prove the conjecture which we present in our introduction.

Assume $Z = Z_0$. Under condition (2.17), we know that $\inf I_\alpha(u) = I_\alpha(u_\alpha)$ for some $u_\alpha \geq 0$, $||u_\alpha||_{p,\Omega} = 1$ and u_α satisfies (2.18). In contrast to the case of $Z < Z_0$, here, we claim:

$$\int_{\partial\Omega} u_\alpha^q \to 0 \quad \text{as} \quad \alpha \to \infty. \tag{4.1}$$

We show (4.1) by contradiction. If not, as $\alpha \to \infty$,

$$\int_{\partial\Omega} u_\alpha^q \geq C > 0. \tag{4.2}$$

Define μ_α, Ω_α, and v_α as in (2.21), then v_α satisfies (2.22). As before, from (2.19) and (4.2), we know $\alpha\mu_\alpha \to 0$. Let $\overline{\lim}_{\alpha\to\infty} dist(x_\alpha, \partial\Omega)/\mu_\alpha = T$. If $T < \infty$, then $v_\alpha \to v_0$ in $C^3(\overline{\Omega_\alpha} \cap B_R(0))$ for all $R > 1$, where v_0 satisfies

$$\begin{cases} -\Delta v_0 = \frac{1}{S_1} v_0^{p-1} & \text{in} \quad \mathbb{R}^n_T, \\ \frac{\partial v_0}{\partial \nu} = -Z_0(\int_{\partial\mathbb{R}^n_T} v_0^q)^{2/q-1} v_0^{q-1} & \text{on} \quad \partial\mathbb{R}^n_T, \\ v_0(0) = 1, \qquad 0 \leq v_0(x) \leq 1. \end{cases} \tag{4.3}$$

However, a similar discussion as in the proof of Theorem 0.1 shows that (4.3) has no solution. Therefore $T = \infty$, and $v_\alpha \to v_1$ in $C^3(\overline{\Omega_\alpha} \cap B_R(0))$ for all $R > 1$, where v_1 satisfies

$$\begin{cases} -\Delta v_1 = \frac{1}{S_1} v_1^{p-1} \quad \text{in} \quad \mathbb{R}^n, \\ v_1(0) = 1, \qquad 0 \leq v_1(x) \leq 1. \end{cases} \tag{4.4}$$

It follows that $||v_1||_{p,\mathbb{R}^n} = 1$, therefore $||v_\alpha - v_1||_{p,\Omega_\alpha} \to 0$.

Note $||\nabla v_\alpha - \nabla v_1||_{2,\Omega_\alpha} < C$. Using Lemma 1.6 and property of v_1, we know that $\int_{\partial\Omega_\alpha} v_\alpha^q \to 0$. This contradicts to (4.2).

This discussion shows that we do need some new ideas to handle the extremal case $Z = Z_0$ in the proof of Theorem 0.2 and 0.3.

5 Appendix

In this appendix, we present another proof of Theorem 0.1 based on a new result due to Carlen and Loss [9].

Proposition 5.1 *(***Carlen and Loss's Theorem***) For* $\lambda > 0$

$$S(\lambda) = \inf_{D^{1,2}(\mathbb{R}^n_+)\setminus\{0\}} \left\{ \frac{||\nabla f||_{2,\mathbb{R}^n_+}}{||f||_{p,\mathbb{R}^n_+}} : \frac{||f||_{q,\partial\mathbb{R}^n_+}}{||f||_{p,\mathbb{R}^n_+}} = \lambda \right\} \tag{5.1}$$

is attained.

Denote $S(0) = 1/S_1^{1/2}$. It is not difficult to see from [9] that $S(\lambda)$ is a continuous function on $[0, \infty)$.

Let $II_Z(u)$ be given as in Section 1 and ξ_Z be given by (1.1). In order to prove Theorem 0.1, we only need to establish the following proposition, the other details can be carried out as in Section 1.

Proposition 5.2 *For any* $Z \in (-1/S, Z_0)$*,* $\inf II_Z$ *is attained.*

Proof. For $Z \leq 0$, this proposition was already proved in [9]. Consequently, a new proof of Escobar's inequality was given by E. Carlen and M. Loss there. Here, we focus on the case of $0 < Z < Z_0$.

It is well known that $\xi_Z \geq 1/(2^{2/n} S_1)$ for $Z \geq 0$. The existence of minimizer of II_Z is equivalent to the existence of a extremal function for the following inequality

$$||\nabla u||^2_{2,\mathbb{R}^n_+} + Z||u||^2_{q,\partial\mathbb{R}^n_+} \geq \xi_Z ||u||^2_{p,\mathbb{R}^n_+}, \quad \forall u \in D^{1,2}(\mathbb{R}^n_+) \setminus \{0\},$$

i.e.

$$||u||^2_{p,\mathbb{R}^n_+} - \frac{Z}{\xi_Z}||u||^2_{q,\partial\mathbb{R}^n_+} \leq \frac{1}{\xi_Z}||\nabla u||^2_{2,\mathbb{R}^n_+}, \quad \forall u \in D^{1,2}(\mathbb{R}^n_+) \setminus \{0\}. \tag{5.2}$$

Therefore, we only need to show

$$\sup_{D^{1,2}(\mathbb{R}^n_+)\setminus\{0\}} \frac{||u||^2_{p,\mathbb{R}^n_+} - Z \cdot \xi_Z^{-1}||u||^2_{q,\partial\mathbb{R}^n_+}}{||\nabla u||^2_{2,\mathbb{R}^n_+}} = \frac{1}{\xi_Z}$$

and the supremum is attained.

From the definition of ξ_Z, it is not difficult to see that the supremum is less than or equals to $1/\xi_Z$. Suppose

$$\sup_{D^{1,2}(\mathbb{R}^n_+)\setminus\{0\}} \frac{||u||^2_{p,\mathbb{R}^n_+} - Z \cdot \xi_Z^{-1}||u||^2_{q,\partial\mathbb{R}^n_+}}{||\nabla u||^2_{2,\mathbb{R}^n_+}} = \frac{1}{\tau\xi_Z}$$

for some $\tau > 1$. Then

$$||u||^2_{p,\mathbb{R}^n_+} \leq \frac{1}{\tau\xi_Z}||\nabla u||^2_{2,\mathbb{R}^n_+} + \frac{Z}{\xi_Z}||u||^2_{q,\partial\mathbb{R}^n_+}, \quad \forall u \in D^{1,2}(\mathbb{R}^n_+) \setminus \{0\}. \tag{5.3}$$

From the definition of ξ_Z, we know that for all $i \geq 1$, there exists u_i, such that

$$||u_i||^2_{p,\mathbb{R}^n_+} \geq \frac{1}{\xi_Z + \frac{1}{i}}(||\nabla u_i||^2_{2,\mathbb{R}^n_+} + Z||u_i||^2_{q,\partial\mathbb{R}^n_+}) \tag{5.4}$$

and $||\nabla u_i||^2_{2,\mathbb{R}^n_+} = 1$. Due to trace inequality, we know $||u_i||_{q,\partial\mathbb{R}^n_+} \leq C$.

Combining (5.3) with (5.4), we have

$$(\frac{1}{\xi_Z + \frac{1}{i}} - \frac{1}{\tau\xi_Z})||\nabla u_i||^2_{2,\mathbb{R}^n_+} \leq (\frac{Z}{\xi_Z} - \frac{Z}{\xi_Z + \frac{1}{i}})||u_i||^2_{q,\partial\mathbb{R}^n_+}.$$

Sending i to infinity, we have $||\nabla u_i||^2_{2,\mathbb{R}^n_+} \to 0$. Contradiction! Therefore $\tau = 1$.

To see the supremum is attained, one observes that

$$\begin{aligned}\sup_{D^{1,2}(\mathbb{R}^n_+)\setminus\{0\}} \frac{||u||^2_{p,\mathbb{R}^n_+} - Z \cdot \xi_Z^{-1}||u||^2_{q,\partial\mathbb{R}^n_+}}{||\nabla u||^2_{2,\mathbb{R}^n_+}} &= \sup\nolimits_{\lambda\geq 0}\{(1 - \xi_Z^{-1} \cdot Z\lambda^2)/S(\lambda)^2 \\ &= \sup\nolimits_{(\xi_Z^{-1}\cdot Z)^{-1/2}\geq\lambda\geq 0}\{(1 - \xi_Z^{-1} \cdot Z\lambda^2)/S(\lambda)^2.\end{aligned}$$

From our early calculation (see (1.22)), we can easily see that $\xi_Z < 1/S_1$, therefore the supremum can not be attained at $\lambda = 0$, that is

$$\sup_{D^{1,2}(\mathbb{R}^n_+)\setminus\{0\}} \frac{||u||^2_{p,\mathbb{R}^n_+} - Z \cdot \xi_Z^{-1}||u||^2_{q,\partial\mathbb{R}^n_+}}{||\nabla u||^2_{2,\mathbb{R}^n_+}} = \sup_{(\xi_Z^{-1}\cdot Z)^{-1/2}\geq\lambda>\gamma} \frac{(1 - \xi_Z^{-1} \cdot Z\lambda^2)}{S(\lambda)^2}$$

for some small $\gamma > 0$. The existence of a maximum follows from the continuity of $S(\lambda)$ and Proposition 5.1.

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