Conformal Curvature Flows on S^1 and Image Processing

Meijun Zhu

University of Oklahoma

Outline

- Setup of the question: introducing curvatures and curvature flow equations.

Outline

- Setup of the question: introducing curvatures and curvature flow equations.
- Outline the main theorems: the steady states, existences and exponential convergence of various flows.

Outline

- Setup of the question: introducing curvatures and curvature flow equations.
- Outline the main theorems: the steady states, existences and exponential convergence of various flows.
- Application in image processing.

Intoduction

The Yamabe problem on a compact Riemannian manifold $(M^n, g_0)(n \geq 3)$ is to find $g \in [g_0]$ such that (M^n, g) has constant scalar curvature.

Intoduction

The Yamabe problem on a compact Riemannian manifold $(M^n, g_0)(n \geq 3)$ is to find $g \in [g_0]$ such that (M^n, g) has constant scalar curvature. Parabolic: The normalized Yamabe flow is defined as

$$\partial_t g = (\overline{R}_g - R_g)g. \tag{1}$$

Intoduction

The Yamabe problem on a compact Riemannian manifold $(M^n, g_0)(n \geq 3)$ is to find $g \in [g_0]$ such that (M^n, g) has constant scalar curvature. Parabolic: The normalized Yamabe flow is defined as

$$\partial_t g = (\overline{R}_g - R_g)g. \tag{1}$$

Elliptic: The conformal Laplacian

$$L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g$$

is a conformal covariant,

Intoduction

The Yamabe problem on a compact Riemannian manifold $(M^n, g_0)(n \geq 3)$ is to find $g \in [g_0]$ such that (M^n, g) has constant scalar curvature. Parabolic: The normalized Yamabe flow is defined as

$$\partial_t g = (\overline{R}_g - R_g)g. \tag{1}$$

Elliptic: The conformal Laplacian

$$L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g$$

is a conformal covariant, namely for any $g_1 \in [g]$, writing $g_1 = \varphi^{\frac{4}{n-2}} g$ with $\varphi > 0$, we have

$$L_{g_1} u = L_{\varphi^{\frac{4}{n-2}} g} u = \varphi^{-\frac{n+2}{n-2}} L_g(\varphi u).$$

Introduction

The conformal Laplacian $L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g$ is not well defined when $n < 3$.

Introduction

The conformal Laplacian $L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g$ is not well defined when $n < 3$.

$n = 2$: On S^2, if $g = e^{2u}g_s$, then $R_g = e^{-2u}(-\Delta_{g_s}u + 2)$ and the conformal Laplacian is given by $L_g u = -2\Delta_g u + R_g$. Related to the uniformization Theorem.

Introduction

The conformal Laplacian $L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g$ is not well defined when $n < 3$.

$n = 2$: On S^2, if $g = e^{2u}g_s$, then $R_g = e^{-2u}(-\Delta_{g_s}u + 2)$ and the conformal Laplacian is given by $L_g u = -2\Delta_g u + R_g$. Related to the uniformization Theorem.

How about $n = 1$? What are the similar problems?

Introduction

The conformal Laplacian $L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g$ is not well defined when $n < 3$.

$n = 2$: On S^2, if $g = e^{2u}g_s$, then $R_g = e^{-2u}(-\Delta_{g_s}u + 2)$ and the conformal Laplacian is given by $L_g u = -2\Delta_g u + R_g$. Related to the uniformization Theorem.

How about $n = 1$? What are the similar problems?

- Curvatures, conformal covariant operators?

Introduction

The conformal Laplacian $L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g$ is not well defined when $n < 3$.

$n = 2$: On S^2, if $g = e^{2u}g_s$, then $R_g = e^{-2u}(-\Delta_{g_s}u + 2)$ and the conformal Laplacian is given by $L_g u = -2\Delta_g u + R_g$. Related to the uniformization Theorem.

How about $n = 1$? What are the similar problems?

- Curvatures, conformal covariant operators?
- Existence of extremal metrics? Parabolic approach?

Introduction

The conformal Laplacian $L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g$ is not well defined when $n < 3$.

$n = 2$: On S^2, if $g = e^{2u}g_s$, then $R_g = e^{-2u}(-\Delta_{g_s}u + 2)$ and the conformal Laplacian is given by $L_g u = -2\Delta_g u + R_g$. Related to the uniformization Theorem.

How about $n = 1$? What are the similar problems?

- Curvatures, conformal covariant operators?
- Existence of extremal metrics? Parabolic approach?
- Global existences and convergence of the flows?

The α-curvature

Let (S^1, g_s) be the unit circle with standard metric $g_s = d\theta \otimes d\theta$. For any metric g on S^1, we write $g := d\sigma \otimes d\sigma = v^{-4} g_s$ for some positive function v and define a general α-curvature of g for any positive constant α by

$$R_g^\alpha = v^3(\alpha v_{\theta\theta} + v).$$

The α-curvature

Let (S^1, g_s) be the unit circle with standard metric $g_s = d\theta \otimes d\theta$. For any metric g on S^1, we write $g := d\sigma \otimes d\sigma = v^{-4} g_s$ for some positive function v and define a general α-curvature of g for any positive constant α by

$$R_g^\alpha = v^3(\alpha v_{\theta\theta} + v).$$

α-conformal Lapalacian of g is defined by:

$$L_g^\alpha = \alpha \Delta_g + R_g^\alpha,$$

where $\Delta_g = D_{\sigma\sigma}$.

The α-curvature

Let (S^1, g_s) be the unit circle with standard metric $g_s = d\theta \otimes d\theta$. For any metric g on S^1, we write $g := d\sigma \otimes d\sigma = v^{-4} g_s$ for some positive function v and define a general α-curvature of g for any positive constant α by

$$R_g^\alpha = v^3(\alpha v_{\theta\theta} + v).$$

α-conformal Lapalacian of g is defined by:

$$L_g^\alpha = \alpha \Delta_g + R_g^\alpha,$$

where $\Delta_g = D_{\sigma\sigma}$. L_g^α is a conformal covariant:

Proposition: For $\varphi > 0$, if $g_2 = \varphi^{-4} g_1$ then $R_{g_2}^\alpha = \varphi^3 L_{g_1}^\alpha \varphi$, and

$$L_{g_2}^\alpha(\psi) = \varphi^3 L_{g_1}^\alpha(\psi\varphi), \quad \forall \psi \in C^2(\mathbf{S^1}).$$

The α-curvature

Existence of an extremal metric in the same class by deformation? Parabolic approach: Introduce α-curvature flow as

$$\partial_t g = (\overline{R}_g^\alpha - R_g^\alpha)g. \tag{2}$$

The α-curvature

Existence of an extremal metric in the same class by deformation? Parabolic approach: Introduce α-curvature flow as

$$\partial_t g = (\overline{R}_g^\alpha - R_g^\alpha) g. \tag{2}$$

Along the flow we have

$$\partial_t \overline{R_g^\alpha} = \frac{1}{4\pi} \int (R_g^\alpha - \overline{R_g^\alpha})^2 d\sigma.$$

The α-curvature

Existence of an extremal metric in the same class by deformation? Parabolic approach: Introduce α-curvature flow as

$$\partial_t g = (\overline{R}_g^\alpha - R_g^\alpha) g. \tag{2}$$

Along the flow we have

$$\partial_t \overline{R_g^\alpha} = \frac{1}{4\pi} \int (R_g^\alpha - \overline{R_g^\alpha})^2 d\sigma.$$

Will see shortly

$$\overline{R_g^\alpha} \leq 1.$$

for some special α.

Affine curvature=1-curvature

Two cases $\alpha = 1$ and $\alpha = 4$ are of special interest.

Affine curvature=1-curvature

Two cases $\alpha = 1$ and $\alpha = 4$ are of special interest.

Case of $\alpha = 1$. The affine curvature of a closed strictly convex curve = its 1-curvature: $\mathbf{x}(\theta) \subset \mathbf{R}^2$ ($\theta \in [0, 2\pi]$) be such a curve, parameterized by the angle θ between the tangent line and x-axis. The affine arc-length function $\sigma(\theta) = \int_0^\theta k^{-2/3} d\theta$, where $k = k(\theta)$ is the curvature. Then the affine curvature of $\mathbf{x}(\theta)$ is given by:

$$\kappa = k((k^{\frac{1}{3}})_{\theta\theta} + k^{\frac{1}{3}}) = (k^{\frac{1}{3}})^3((k^{\frac{1}{3}})_{\theta\theta} + k^{\frac{1}{3}}).$$

It coincides with 1-curvature $R^1_{g_x}$, where $g_x = (k^{\frac{1}{3}})^{-4} d\theta \otimes d\theta$.

Affine curvature =1-curvature

One the other hand, given (S^1, g), write $g = u^{-4} g_s$. Suppose $u(\theta)$ satisfies the orthogonal condition:

$$\int_0^{2\pi} \frac{\cos\theta}{u^3(\theta)} d\theta = \int_0^{2\pi} \frac{\sin\theta}{u^3(\theta)} d\theta = 0. \quad (3)$$

Define $\mathbf{x}(\theta)$ as

$$\mathbf{x}(\theta) = \left(\int_0^{\theta} \frac{\cos\theta}{u^3(\theta)} d\theta, \int_0^{\theta} \frac{\sin\theta}{u^3(\theta)} d\theta \right).$$

Then $\mathbf{x}(\theta)$ is a closed strictly convex curve, and its affine curvature is equal to R_g^1.

The α-curvature

Therefore we have the following correspondece:

$$\left[\begin{array}{c}\text{Close convex curve } \mathbf{x}(\theta) \\ \text{with curvature } k(\theta). \\ \text{Affine curvature } \kappa(\theta)\end{array}\right] \leftrightarrow \left[\begin{array}{c}\text{Metric } g = (k^{\frac{1}{3}})^{-4} g_s \text{ on } S^1 \\ \text{with } v = k^{\frac{1}{3}} \text{ satisfies (3).} \\ \text{1-curvature } R_g^1\end{array}\right]$$

The α-curvature

Therefore we have the following correspondece:

$$\begin{bmatrix} \text{Close convex curve } \mathbf{x}(\theta) \\ \text{with curvature } k(\theta). \\ \text{Affine curvature } \kappa(\theta) \end{bmatrix} \leftrightarrow \begin{bmatrix} \text{Metric } g = (k^{\frac{1}{3}})^{-4} g_s \text{ on } S^1 \\ \text{with } v = k^{\frac{1}{3}} \text{ satisfies (3).} \\ \text{1-curvature } R_g^1 \end{bmatrix}$$

$\alpha = 4$: The 4-curvature R_g^4 can be viewed as the scalar curvature in an analogous one-dimensional Yamabe flow.

The α-curvature

Therefore we have the following correspondece:

$$\begin{bmatrix} \text{Close convex curve } \mathbf{x}(\theta) \\ \text{with curvature } k(\theta). \\ \text{Affine curvature } \kappa(\theta) \end{bmatrix} \leftrightarrow \begin{bmatrix} \text{Metric } g = (k^{\frac{1}{3}})^{-4} g_s \text{ on } S^1 \\ \text{with } v = k^{\frac{1}{3}} \text{ satisfies (3).} \\ \text{1-curvature } R_g^1 \end{bmatrix}$$

$\alpha = 4$: The 4-curvature R_g^4 can be viewed as the scalar curvature in an analogous one-dimensional Yamabe flow.

NOW: flow method to find the extremal metrics for the cases $\alpha = 1$ and $\alpha = 4$. We denote R_g^1 by κ.

1-curvature flow

An analytic proof to the following general Blaschke-Santaló type inequality, which implies that $\overline{\kappa}_g$ is bounded above, and classifies all the extremal metrics:

General Blaschke-Santaló inequality

For $u(\theta) \in H^1(S^1)$ *and* $u > 0$, *if* u *satisfies the orthogonal condition (3), then*

$$\int_0^{2\pi} (u^2 - u_\theta^2) d\theta \int_0^{2\pi} u^{-2}(\theta) d\theta \leq 4\pi^2,$$

1-curvature flow

An analytic proof to the following general Blaschke-Santaló type inequality, which implies that $\overline{\kappa}_g$ is bounded above, and classifies all the extremal metrics:

General Blaschke-Santaló inequality

For $u(\theta) \in H^1(S^1)$ *and* $u > 0$, *if* u *satisfies the orthogonal condition (3), then*

$$\int_0^{2\pi} (u^2 - u_\theta^2)d\theta \int_0^{2\pi} u^{-2}(\theta)d\theta \leq 4\pi^2,$$

and the equality holds if and only if

$$u(\theta) = c\sqrt{\lambda^2 \cos^2(\theta - \alpha) + \lambda^{-2} \sin^2(\theta - \alpha)}.$$

Exponential convergence

Theorem: *Suppose* $g_0 = u^{-4}(\theta, 0) g_s$ *and* $u(\theta, 0)$ *satisfies the orthogonal condition (3). Then there is a unique smooth solution to the flow equation*

$$\partial_t g = (\overline{\kappa} - \kappa) g, \quad g(\theta, 0) = g_0(\theta)$$

for $t \in [0, +\infty)$. *Moreover,* $g(t) \to g_\infty$ *exponentially as* $t \to +\infty$, *and the 1-curvature of* g_∞ *is constant.*

Exponential convergence

Theorem: *Suppose* $g_0 = u^{-4}(\theta, 0) g_s$ *and* $u(\theta, 0)$ *satisfies the orthogonal condition (3). Then there is a unique smooth solution to the flow equation*

$$\partial_t g = (\overline{\kappa} - \kappa) g, \quad g(\theta, 0) = g_0(\theta)$$

for $t \in [0, +\infty)$. *Moreover,* $g(t) \to g_\infty$ *exponentially as* $t \to +\infty$, *and the 1-curvature of* g_∞ *is constant.*

Our 1-curvature flow ⇔ The normalized affine curve-shortening flow,

$$\mathbf{x}_t(\sigma, t) = \mathbf{x}_{\sigma\sigma}.$$

Outline of the proof

Along the flow $\partial_t g = (\overline{\kappa}_g - \kappa_g)g$, we have

$$\kappa_t = \frac{1}{4}\Delta\kappa + \kappa(\kappa - \overline{\kappa}), \qquad u_t = \frac{1}{4}(\kappa - \overline{\kappa})u,$$

i.e.

$$u_t = \frac{1}{4}u^4\Delta_{g_s}u + \frac{1}{4}u^5 - \frac{1}{4}\overline{\kappa}u.$$

Outline of the proof

Along the flow $\partial_t g = (\overline{\kappa}_g - \kappa_g)g$, we have

$$\kappa_t = \frac{1}{4}\Delta\kappa + \kappa(\kappa - \overline{\kappa}), \qquad u_t = \frac{1}{4}(\kappa - \overline{\kappa})u,$$

i.e.

$$u_t = \frac{1}{4}u^4\Delta_{g_s}u + \frac{1}{4}u^5 - \frac{1}{4}\overline{\kappa}u.$$

Therefore

$$\partial_t \int d\sigma = \partial_t \int u^{-2}d\theta = \int \frac{1}{2}(\overline{\kappa} - \kappa)d\sigma = 0,$$

$$\partial_t \int_0^{2\pi} u^{-3}(\theta)\cos\theta d\theta = \frac{3\overline{\kappa}}{4}\int_0^{2\pi} u^{-3}(\theta)\cos\theta d\theta.$$

It preserves the length and the orthogonality!

Existence of the flow

Suppose $g(t)$ satisfies the flow equation on $[0, T]$, then we have

$$\kappa_t + \overline{\kappa}\kappa = \frac{1}{4}\Delta\kappa + \kappa^2 \geq \frac{1}{4}\Delta\kappa.$$

Existence of the flow

Suppose $g(t)$ satisfies the flow equation on $[0, T]$, then we have

$$\kappa_t + \overline{\kappa}\kappa = \frac{1}{4}\Delta\kappa + \kappa^2 \geq \frac{1}{4}\Delta\kappa.$$

It follows from the maximum principle that

$$\kappa \geq \min \kappa(\sigma, 0) e^{-\int_0^t \overline{\kappa} d\tau},$$

then: $u(\sigma, t) = u(\sigma, 0) \cdot e^{\frac{1}{4}\int_0^t (\kappa - \overline{\kappa}) d\tau} \geq \tilde{c}_2(\kappa(\sigma, 0), t_0) > 0.$

Existence of the flow

Suppose $g(t)$ satisfies the flow equation on $[0, T]$, then we have

$$\kappa_t + \overline{\kappa}\kappa = \frac{1}{4}\Delta\kappa + \kappa^2 \geq \frac{1}{4}\Delta\kappa.$$

It follows from the maximum principle that

$$\kappa \geq \min \kappa(\sigma, 0) e^{-\int_0^t \overline{\kappa} d\tau},$$

then: $u(\sigma, t) = u(\sigma, 0) \cdot e^{\frac{1}{4}\int_0^t (\kappa - \overline{\kappa}) d\tau} \geq \tilde{c}_2(\kappa(\sigma, 0), t_0) > 0.$

Using the evolution equation of u and the orthogonal condition, we can prove that there exist $C = C(T)$ such that $\frac{1}{C} \leq u \leq C$. Therefore, from standard parabolic estimates that the solution exists for all $t \in [0, \infty)$.

Convergence of the flow

For $p \geq 2$, we define

$$F_p(t) = \int_0^{2\pi} |\kappa - \overline{\kappa}|^p d\sigma = ||\kappa - \overline{\kappa}||_{L^p}^p.$$

Convergence of the flow

For $p \geq 2$, we define

$$F_p(t) = \int_0^{2\pi} |\kappa - \overline{\kappa}|^p d\sigma = ||\kappa - \overline{\kappa}||_{L^p}^p.$$

Direct computation yields

$$\partial_t F_p \leq C_1(p)(F_{p+1} + F_p + F_p^{1+\frac{1}{p}}) - C_2(p) \int_0^{2\pi} (|\kappa - \overline{\kappa}|^{\frac{p}{2}})_\sigma^2 d\sigma.$$

Convergence of the flow

For $p \geq 2$, we define

$$F_p(t) = \int_0^{2\pi} |\kappa - \overline{\kappa}|^p d\sigma = ||\kappa - \overline{\kappa}||_{L^p}^p.$$

Direct computation yields

$$\partial_t F_p \leq C_1(p)(F_{p+1} + F_p + F_p^{1+\frac{1}{p}}) - C_2(p) \int_0^{2\pi} (|\kappa - \overline{\kappa}|^{\frac{p}{2}})_\sigma^2 d\sigma.$$

Using Sobolev inequality and Young's inequality we have

$$\partial_t F_p \leq C_2(p)(F_p + F_p^\beta + F_p^{1+\frac{1}{p}}) - C_3(p) F_{3p}^{\frac{1}{3}},$$

where $\beta = \frac{2p-1}{2p-3} > 1$ and C_2, C_3 are positive constants.

Convergence of the flow

Since $\partial_t \overline{\kappa}_g = \frac{1}{4\pi} \int (\kappa_g - \overline{\kappa}_g)^2 d\sigma$ and $\overline{\kappa}$ is bounded above, we have

$$\int_0^\infty F_2(t) dt < \infty.$$

Convergence of the flow

Since $\partial_t \overline{\kappa}_g = \frac{1}{4\pi} \int (\kappa_g - \overline{\kappa}_g)^2 d\sigma$ and $\overline{\kappa}$ is bounded above, we have

$$\int_0^\infty F_2(t)dt < \infty.$$

Using above two inequalities, Hölder's inequality and induction, we can prove the following Lemma:

Lemma: For any $p \geq 2$,

$$F_p(t) \to 0 \ \text{ as } \ t \to +\infty \quad \text{and} \quad \int_0^\infty F_p(t)dt < \infty.$$

Convergence of the flow

Since $\partial_t \overline{\kappa}_g = \frac{1}{4\pi} \int (\kappa_g - \overline{\kappa}_g)^2 d\sigma$ and $\overline{\kappa}$ is bounded above, we have

$$\int_0^\infty F_2(t)dt < \infty.$$

Using above two inequalities, Hölder's inequality and induction, we can prove the following Lemma:

Lemma: For any $p \geq 2$,

$$F_p(t) \to 0 \text{ as } t \to +\infty \quad \text{and} \quad \int_0^\infty F_p(t)dt < \infty.$$

Then we estimate $\int_0^{2\pi} (\kappa_\sigma)^2 d\sigma$ and obtain that

$$||\kappa - \bar{\kappa}||_{L^\infty} \to 0, \text{ as } t \to \infty.$$

Exponential Convergence

To show exponential convergence:

$$\partial_t F_2 \leq (-\frac{1}{2} + o(1))F_2,$$

which implies $F_2 \leq Ce^{-at}$ for some $C,\ a > 0$.

Exponential Convergence

To show exponential convergence:

$$\partial_t F_2 \leq (-\frac{1}{2} + o(1))F_2,$$

which implies $F_2 \leq Ce^{-at}$ for some C, $a > 0$. Then from the flow equation

$$u_t = \frac{1}{4}(\kappa - \overline{\kappa})u,$$

we can prove that $u(t)$ converges exponentially to some u_∞ as $t \to \infty$:

$$||u(t) - u_\infty||_{L^\infty} \leq Ce^{-at/2},$$

and the 1-curvature of $g_\infty := u_\infty^{-4} g_s$ is constant 1. This completes the proof of our main Theorem.

Theorem ($\alpha = 4$)

For $\alpha = 4$, we have the similar theorem:

Theorem: For an abstract curve $(S^1, {u_0}^{-4} g_s)$, then there is a unique smooth solution to the flow equation

$$\partial_t g = (\overline{R}_g^4 - R_g^4) g, \quad g(\theta, 0) = g_0(\theta)$$

for $t \in [0, +\infty)$. Moreover, $g(t) \to g_\infty$ exponentially as $t \to +\infty$, and the 4-curvature of g_∞ is constant.

Theorem ($\alpha = 4$)

For $\alpha = 4$, we have the similar theorem:

Theorem: For an abstract curve $(S^1, {u_0}^{-4} g_s)$, then there is a unique smooth solution to the flow equation

$$\partial_t g = (\overline{R}_g^4 - R_g^4) g, \quad g(\theta, 0) = g_0(\theta)$$

for $t \in [0, +\infty)$. Moreover, $g(t) \to g_\infty$ exponentially as $t \to +\infty$, and the 4-curvature of g_∞ is constant.

Along the flow $\overline{R}_g^4$ is always increasing,

$$\partial_t \overline{R}_g^4 = \frac{1}{4\pi} \int (R_g^4 - \overline{R}_g^4)^2 d\sigma.$$

Upper bound for ($\alpha = 4$)

The upper unbound of $\overline{R}_g^4$ follows from the following inequality:

Proposition: For $u(\theta) \in H^1(S^1)$ and $u > 0$,

$$\int_0^{2\pi} (\frac{1}{4}u^2 - u_\theta^2) d\theta \int_0^{2\pi} u^{-2}(\theta) d\theta \leq \pi^2,$$

and the equality holds if and only if

$$u(\theta) = c\sqrt{\lambda^2 \cos^2 \frac{\theta - \alpha}{2} + \lambda^{-2} \sin^2 \frac{\theta - \alpha}{2}},$$

for some $\lambda, c > 0$ and $\alpha \in [0, 2\pi)$.

Q-curvature

For any given g on S^1 we write $g = v^{-4/3} g_s$, where g_s is the standard metric. We define general α-Q-curvature on (S^1, g) as

$$Q_g^\alpha = v^{\frac{5}{3}}\left(\frac{\alpha^2}{9} v_{\theta\theta\theta\theta} + \frac{10\alpha}{9} v_{\theta\theta} + v\right),$$

Q-curvature

For any given g on S^1 we write $g = v^{-4/3} g_s$, where g_s is the standard metric. We define general α-Q-curvature on (S^1, g) as

$$Q_g^\alpha = v^{\frac{5}{3}}(\frac{\alpha^2}{9} v_{\theta\theta\theta\theta} + \frac{10\alpha}{9} v_{\theta\theta} + v),$$

and the corresponding operator as

$$P_g^\alpha(f) = \frac{\alpha^2}{9}\Delta_g^2 f + \frac{10\alpha}{9}\nabla_g(R_g^\alpha \nabla_g f) + Q_g^\alpha f,$$

where R_g^α is the α-curvature of (S^1, g).

Q-curvature

For any given g on S^1 we write $g = v^{-4/3} g_s$, where g_s is the standard metric. We define general α-Q-curvature on (S^1, g) as

$$Q_g^\alpha = v^{\frac{5}{3}}(\frac{\alpha^2}{9} v_{\theta\theta\theta\theta} + \frac{10\alpha}{9} v_{\theta\theta} + v),$$

and the corresponding operator as

$$P_g^\alpha(f) = \frac{\alpha^2}{9}\Delta_g^2 f + \frac{10\alpha}{9}\nabla_g(R_g^\alpha \nabla_g f) + Q_g^\alpha f,$$

where R_g^α is the α-curvature of (S^1, g).

P_g^α is a conformal covariant: if $g_2 = \varphi^{-\frac{4}{3}} g_1$, then $Q_{g_2}^\alpha = \varphi^{\frac{5}{3}} P_{g_1}^\alpha \varphi$ and $P_{g_2}^\alpha \psi = \varphi^{\frac{5}{3}} P_{g_1}^\alpha(\psi\varphi), \forall \psi$.

Q-curvature flows

Existence of extremal metrics?

Introduce α-Q-curvature flow

$$\partial_t g = (Q_g^\alpha - \overline{Q}_g^\alpha) g.$$

Q-curvature flows

Existence of extremal metrics?

Introduce α-Q-curvature flow

$$\partial_t g = (Q_g^\alpha - \overline{Q}_g^\alpha) g.$$

In such a setting, the above flow is again a gradient flow of $\overline{Q}_g^\alpha$:
Along the flow, $\overline{Q}_g^\alpha$ is always decreasing:

$$\partial_t \overline{Q}_g^\alpha = -\frac{3}{4\pi} \int_{S^1} (Q_g^\alpha - \overline{Q}_g^\alpha)^2 d\sigma.$$

Q-curvature flows

Existence of extremal metrics?

Introduce α-Q-curvature flow

$$\partial_t g = (Q_g^\alpha - \overline{Q}_g^\alpha) g.$$

In such a setting, the above flow is again a gradient flow of $\overline{Q}_g^\alpha$: Along the flow, $\overline{Q}_g^\alpha$ is always decreasing:

$$\partial_t \overline{Q}_g^\alpha = -\frac{3}{4\pi} \int_{S^1} (Q_g^\alpha - \overline{Q}_g^\alpha)^2 d\sigma.$$

Again, we are interested in two cases: $\alpha = 1$ and $\alpha = 4$.

Very recently, we prove the existence and convergence of the α-Q-curvature flow in these two cases.

1-Q-curvature flow

Hard part: the proof to the following inequality, which implies that $\overline{Q_g^1}$ is bounded from below, and classifies all the extremal metrics:

Blaschke-Santaló inequality involving higher order derivative?

For $u(\theta) \in H^2(S^1)$ *and* $u > 0$, *satisfying the orthogonal condition*

$$\int_0^{2\pi} \frac{\cos^3\theta}{u^{5/3}(\theta)} d\theta = \int_0^{2\pi} \frac{\sin^3\theta}{u^{5/3}(\theta)} d\theta = 0, \tag{4}$$

$$\int_0^{2\pi} (u_{\theta\theta}^2 - 10u_\theta^2 + 9u^2) d\theta \left(\int_0^{2\pi} u^{-2/3}(\theta) d\theta \right)^3 \geq 144\pi^4,$$

1-Q-curvature flow

Hard part: the proof to the following inequality, which implies that $\overline{Q^1_g}$ is bounded from below, and classifies all the extremal metrics:

Blaschke-Santaló inequality involving higher order derivative?

For $u(\theta) \in H^2(S^1)$ *and* $u > 0$, *satisfying the orthogonal condition*

$$\int_0^{2\pi} \frac{\cos^3\theta}{u^{5/3}(\theta)} d\theta = \int_0^{2\pi} \frac{\sin^3\theta}{u^{5/3}(\theta)} d\theta = 0, \qquad (4)$$

$$\int_0^{2\pi} (u_{\theta\theta}^2 - 10u_\theta^2 + 9u^2) d\theta \left(\int_0^{2\pi} u^{-2/3}(\theta) d\theta \right)^3 \geq 144\pi^4,$$

"=" holds if and only if

$$u_0(\theta) = c\left(\lambda^2\cos^2(\theta-\beta) + \lambda^{-2}\sin^2(\theta-\beta)\right)^{\frac{3}{2}}.$$

Exponential convergence

Theorem: *Suppose the initial metric* $g_0 = v^{-\frac{4}{3}}(\theta, 0)g_s$ *on* S^1 *satisfies the orthogonal condition (4). Then there is a unique smooth solution to the flow equation*

$$\partial_t g = (\overline{Q_g^1} - Q_g^1)g, \quad g(0) = g_0$$

for $t \in [0, +\infty)$. *Moreover,* $g(t) \to g_\infty$ *exponentially as* $t \to +\infty$, *and the 1-*Q*-curvature of* g_∞ *is constant.*

Q-curvature flows

Direct computation shows that

$$Q_g^\alpha = \frac{\alpha}{3}\Delta_g R_g^\alpha + (R_g^\alpha)^2,$$

which implies

$$\int Q_g^\alpha d\sigma = \int (R_g^\alpha)^2 d\sigma.$$

Q-curvature flows

Direct computation shows that

$$Q_g^\alpha = \frac{\alpha}{3}\Delta_g R_g^\alpha + (R_g^\alpha)^2,$$

which implies

$$\int Q_g^\alpha d\sigma = \int (R_g^\alpha)^2 d\sigma.$$

Therefore $\int (R_g^\alpha)^2 d\sigma$ is decreasing along the α-Q-curvature flow. Thus the α-Q-curvature flow can be viewed as one dimensional Calabi flow. **Caution:** It is not trivial to see that $\int (R_g^\alpha)^2 d\sigma$ is bounded below by a positive constant.

Applications in image processing

Applications in image processing

Top left: original image (with impulse noise).

Top right: affine image flow (10 steps).

Bottom left: affine image flow (20 steps).

Bottom right: our 4-th order image flow(10 steps).

Equation for affine flow:

$$\Phi_t = (\Phi_x^2 \Phi_{yy} + \Phi_y^2 \Phi_{xx} - 2\Phi_x \Phi_y \Phi_{xy})^{\frac{1}{3}}, \quad \Phi(0) = I(x, y).$$

How does it work?

Consider contour curve for any $C \geq 0$:

$$\Phi(x, y) = C.$$

Move such curves simultaneously:

$$\Phi((x(t), y(t), t) = C.$$

So

$$\Phi_x \cdot x_t + \Phi_y \cdot y_t + \Phi_t = 0,$$

i.e.

$$\Phi_t = -(x_t, y_t) \cdot (\Phi_x, \Phi_y).$$

curve shortening flow

Let $F(t) = (x(t), y(t))$ represent the curve.

Curve shortening flow is defined by

$$F_t = kN = -k(\frac{\Phi_x}{|\nabla\Phi|}, \frac{\Phi_y}{|\nabla\Phi|}).$$

$\Rightarrow$

$$\begin{cases} \Phi_t = \frac{\Phi_x^2\Phi_{yy}+\Phi_y^2\Phi_{xx}-2\Phi_x\Phi_y\Phi_{xy}}{|\nabla\Phi|^2}, \quad \text{on}[0,+\infty)\times\Omega \\ \Phi(x,y,0) = I(x,y), \ \text{I(x, y) initial image} \\ \text{various boundary condition.} \end{cases}$$

Affine curve shortening flow

Affine shortening flow is defined by

$$F_t = k^{1/3} N = -k^{1/3}\left(\frac{\Phi_x}{|\nabla\Phi|}, \frac{\Phi_y}{|\nabla\Phi|}\right).$$

$\Rightarrow$

$$\begin{cases} \Phi_t = (\Phi_x^2 \Phi_{yy} + \Phi_y^2 \Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy})^{1/3}, \quad \text{on}[0, +\infty) \\ \Phi(x, y, 0) = I(x, y), \ \text{I(x, y) initial image} \\ \text{various boundary condition.} \end{cases}$$

Thank you!

Total curvature

For fixed length ($\int_0^{2\pi} u^{-2} d\theta = 2\pi$), B-S inequality $\Longrightarrow$

$$
\begin{aligned}
2\pi &\geq \int_0^{2\pi} (u^2 - u_\theta^2) d\theta \\
&= \int_0^{2\pi} u^3 (u + u_{\theta\theta}) u^{-2} d\theta \\
&= \int_0^{2\pi} \kappa d\sigma.
\end{aligned}
$$

Thus:

$$\overline{\kappa} \leq 1.$$

Affine isoperimetric inequality

B-S inequality $\Longrightarrow$ for $h > 0$,

$$4\pi^2 \int_0^{2\pi} h(h + h_{\theta\theta})d\theta \geq [\int_0^{2\pi} (h + h_{\theta\theta})^{\frac{2}{3}} d\theta]^3.$$

Let $h =< X, -N >$ be the supporting function of the closed strictly convex curve $X(\theta)$, then $h + h_{\theta\theta} = 1/k$, $\int_0^{2\pi} (h + h_{\theta\theta})^{\frac{2}{3}} d\theta = \int_0^{2\pi} k^{-2/3} d\theta = \sigma$.
Also $\int_0^{2\pi} h(h + h_{\theta\theta})d\theta = 2A$.
So

$$8\pi^2 A \geq \sigma^3.$$