Liouville theorems on some indefinite equations

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Abstract

In this note, we present some Liouville type theorems about the nonnegative solutions to some indefinite elliptic equations.

1 Introduction

In the study of some indefinite elliptic problems, in order to get a priori estimate one may use standard blowup argument. In this procedure, we encounter the following equations in $\mathbb{R}^n$:

$$\Delta u + x_n u^p = 0, \quad u \geq 0, \quad \text{in } \mathbb{R}^n \tag{1.1}$$

where we write $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. Throughout this paper, we always assume $n \geq 3$.

Using some test functions on S^n, Berestycki, Capuzzo Dolcetta and Nirenberg [1] proved the following theorem (we state a weak version in the whole space).

Theorem A. Let $u(x) \in C^2(\mathbb{R}^n)$ be a solution to (1.1). If $p < \frac{n+2}{n-1}$, then $u(x) = 0$.

One observes that $\frac{n+2}{n-1}$ is less than the critical Sobolev exponent $\frac{n+2}{n-2}$. In the study of prescribing scalar curvature problems, we do need to consider those equations with critical exponents. The main purpose of this paper is to study (1.1) for $p = \frac{n+2}{n-2}$.

We here consider a slightly general equation:

$$\Delta u + x_n^k u^p = 0, \quad u \geq 0, \quad \text{in } \mathbb{R}^n \tag{1.2}$$

where and throughout this paper we assume that k is an odd positive integer.

Our first result can be stated as the following.

Theorem 1.1 *Let $u(x) \in C^2(\mathbb{R}^n)$ be a solution to (1.2). If $p = (n+2+2k)/(n-2)$, then $u = 0$.*

Remark 1.1 *It is obvious to see that Theorem 1.1 still holds if we change x_n to x_i for $i = 1, ..., n-1$ in equation (1.2).*

For the critical exponent, we have the following.

Theorem 1.2 *Let $u(x) \in C^2(\mathbb{R}^n)$ be a solution to (1.2). If $p = (n+2)/(n-2)$ and the dimension n is an even number, then $u = 0$.*

Remark 1.2 *For $k \geq 3$, Theorem 1.2 was included in Theorem 2.2 of [1].*

We also consider a related problem in the half space with Neumann boundary condition:

$$\begin{cases} \Delta u = 0, \quad u \geq 0, & \text{in } \mathbb{R}^n_+ \\ \frac{\partial u}{\partial x_n} = x_i^k u^q, & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{1.3}$$

where and through this paper we write $\mathbb{R}^n_+ = \{(x', x_n) = (x_1, ..., x_{n-1}, x_n) \mid x' \in \mathbb{R}^{n-1}, \quad x_n > 0\}$, $i \leq n-1$. This equation can be viewed as the limit equation when we "blow up" some equations with indefinite boundary nonlinearities.

We have the following result.

Theorem 1.3 *Let $u \in C^2(\mathbb{R}^n_+)$ be a solution to (1.3). If $q = (n+2k)/(n-2)$, then $u(x)$ just depends on x_n and x_i.*

Similarly, for the critical exponent, we have the following.

Theorem 1.4 *Let $u \in C^2(\mathbb{R}^n_+)$ be a solution to (1.3). If $q = n/(n-2)$, then $u(x)$ just depends on x_n and x_i.*

An interesting consequence of Theorem 1.4 is the following corollary.

Corollary 1.1 *Let $u \in C^2(\mathbb{R}^n_+)$ be a solution to (1.3). If $q = n/(n-2)$ and $u(x) \to 0$ as $|x| \to \infty$, then $u(x) = 0$.*

Remark 1.3 *We tend to believe that Corollary 1.1 holds without the decay assumption at infinity on $u(x)$. We hope to clarify this point in our future study.*

The proofs of Theorem 1.1 and 1.3 follow from the standard moving plane method, see for instance, [2], [3], [5] in the whole $\mathbb{R}^n$ and [7]-[11] in the upper half space. Here we first observe that the equations with those exponents in Theorem 1.1 and 1.3 are invariant under the Kelvin transformation, thus after we perform the Kelvin transformation on these equations, the coefficients are still monotone in any direction perpendicular to x_n-axis. We then apply the moving plane method as usual in these directions. The main difficulty will come from the analysis of the possible singular point. By adding dimensions, we can prove Theorem 1.2 and 1.4 as in the work of [9]. As an application, we state an existence result concerning the same equations as in [1] in the last section.

2 Proofs of Theorem 1.1 and 1.2

We first derive Theorem 1.2 from Theorem 1.1 by using the method of adding dimensions, which was introduced by us in [9].

Let $u(x) \geq 0$ be a solution to (1.2). Set $\tilde{u}(x_1, x_2, ..., x_n, x_{n+1}, ..., x_m) = u(x_1, ..., x_n)$, where m is an integer which will be chosen later. Thus $\tilde{u}$ solves

$$\Delta \tilde{u} + x_n^k \tilde{u}^{\frac{n+2}{n-2}} = 0, \quad \tilde{u} \geq 0, \quad \text{in } \mathbb{R}^m.$$

Choosing $m = (n-2)(2+k)/2 + 2$ (here we use the fact that n is even), we have $(n+2)/(n-2) = (m+2+2k)/(m-2)$. It follows from Theorem 1.1 that $\tilde{u} = 0$, therefore, $u = 0$.

We now focus on the proof of Theorem 1.1. Later on we write $x = (x', x_n)$ and assume that $u(x)$ solves (1.2). From the strong maximum principle, we know that either $u = 0$ or $u > 0$. We prove $u = 0$ by contradiction. Suppose $u > 0$, we aim to derive a contradiction.

Since there is no assumption on the decay rate of u at infinity, as usual, we set

$$v(x) = \frac{1}{|x|^{n-2}} u(\frac{x}{|x|^2}). \tag{2.4}$$

Then $v(x)$ satisfies

$$\Delta v + x_n^k v^{\frac{n+2+2k}{n-2}} = 0, \quad v > 0, \quad \text{in } \mathbb{R}^n \setminus \{0\}. \tag{2.5}$$

Our purpose is to obtain some symmetric properties about $v(x)$ on the x'-hyperplane. We achieve this by using moving planes which are parallel to x_n-axis. Without loss of generality, we move the planes along x_1-direction.

Our first lemma will be used to handle the possible singular point of $v(x)$ at the origin.

Lemma 2.1 *Assume that v satisfies*

$$\begin{cases} \Delta v + x_n^k v^{\frac{n+2+2k}{n-2}} = 0 & in \ \ B_{1/2} \setminus \{0\} \\ v > 0, \quad v \in C^2(B_{1/2} \setminus \{0\}). \end{cases}$$

If $v(x) \geq \epsilon$ on $\partial B_{1/2}$ for some $\epsilon \leq 1$, then $v(x) \geq \epsilon/2$ in $B_{1/2} \setminus \{0\}$.

Proof: Let $\varphi_1(x) = \frac{\epsilon}{2} - \frac{r^{n-2}\epsilon}{|x|^{n-2}} + \frac{x_n^2 \epsilon}{2}$ in $B_{1/2} \setminus B_r$ for some small $r > 0$, $A(x) = v(x) - \varphi_1(x)$, then

$$\Delta A(x) = -x_n^k v^{\frac{n+2+2k}{n-2}} - \epsilon.$$

Set

$$S = \{x \mid -x_n^k v^{\frac{n+2+2k}{n-2}} - \epsilon > 0, \ \ x \in B_{1/2} \setminus B_r\},$$
$$S^c = \{x \mid -x_n^k v^{\frac{n+2+2k}{n-2}} - \epsilon \leq 0, \ \ x \in B_{1/2} \setminus B_r\}.$$

In $\bar{S}$, $-x_n^k v^{\frac{n+2+2k}{n-2}} - \epsilon \geq 0$, thus $x_n < 0$. It follows that (notice that k is odd)

$$v^{\frac{n+2+2k}{n-2}} \geq \frac{\epsilon}{-x_n^k} \geq \frac{\epsilon}{(1/2)^k} > \varphi_1.$$

Since $\frac{n+2+2k}{n-2} > 1$ and $1 > \epsilon > \varphi_1$ in $\bar{S}$, we know that $v > \varphi_1$ for $x \in \bar{S}$.

In S^c, we know $\Delta A \leq 0$. Also we can check that: On $\{x \mid -x_n^k v^{\frac{n+2+2k}{n-2}} - \epsilon = 0\}$, as in S, $v > \varphi_1$; On $\partial B_{1/2}$, $v - \varphi_1 \geq \epsilon - (\epsilon/2 + \epsilon/8) > 0$; On ∂B_r, $v - \varphi_1 \geq 0 - (\epsilon/2 - \epsilon + \epsilon/8) > 0$; That is : on ∂S^c, $v \geq \varphi_1$. By the maximum principle, we have that $v \geq \varphi_1$ in S^c.

Therefore, we know $v \geq \varphi_1$ in $B_{1/2} \setminus B_r$. Sending $r \to 0$, we complete the proof of the lemma.

Now we are ready to move the planes.

For $\lambda < 0$ we define

$$\Sigma_\lambda = \{x \mid \ x_1 > \lambda\}, \ T_\lambda = \{x \mid \ x_1 = \lambda\},$$
$$\tilde{\Sigma}_\lambda = \overline{\Sigma}_\lambda \setminus \{0\}, \ \ x^\lambda \text{ is the reflection of } x \text{ about } T_\lambda,$$
$$v_\lambda(x) = v(x^\lambda), \ w_\lambda(x) = v(x) - v_\lambda(x).$$

Then $w_\lambda(x)$ satisfies

$$\Delta w_\lambda + x_n^k c(x) w_\lambda = 0 \quad \text{in} \quad \tilde{\Sigma}_\lambda, \tag{2.6}$$

where $c(x) = \frac{n+2+2k}{n-2} \xi^{\frac{2k+4}{n-2}}(x)$, $\xi(x)$ is a positive function between $v(x)$ an $v_\lambda(x)$.

Proposition 2.1 *There exists $R > 1$ such that, if $\lambda < -R$, $w_\lambda \geq 0$ in $\tilde{\Sigma}_\lambda$.*

Proof: As in [8], we choose an auxiliary function $g(x) = |x|^{-\alpha}$ with $0 < \alpha < n-2$, and consider $\bar{w}_\lambda = w_\lambda/g$.

Claim: There exists $R > 1$ such that, if $\lambda < -R$, $\bar{w}_\lambda \geq 0$ in $\tilde{\Sigma}_\lambda$.

Before we prove the claim, we first take care of the possible singular point of w_λ at the origin. Due to the fact that $v(x) \to 0$ as $|x| \to \infty$, we easily see that there exists a $R_1 > 0$ such that if $\lambda < -R_1$, $v_\lambda(x) \leq \frac{1}{4}\min\{\min_{\partial B_{1/2}(0)} v(x), 1\}$ in $B_{1/2}\backslash\{0\}$. Therefore, from Lemma 2.1, we know that $w_\lambda(x) \geq 0$ in $B_{1/2}\backslash\{0\}$ for $\lambda < -R_1$, so is $\bar{w}_\lambda$.

We then prove the claim by contradiction. Assume for any $\lambda < -R_1$, $\inf_{\tilde{\Sigma}_\lambda} \bar{w}_\lambda(x) < 0$. From the above argument and the fact that $v(x) \to 0$ as $|x| \to \infty$, we know that there exists a $\bar{x} \in \tilde{\Sigma}_\lambda$ such that $\bar{w}_\lambda(\bar{x}) = \inf_{\tilde{\Sigma}_\lambda} \bar{w}_\lambda(x) < 0$.

Direct computation shows that $\bar{w}_\lambda$ satisfies

$$\Delta \bar{w}_\lambda + \frac{2}{g}\nabla g \cdot \nabla \bar{w}_\lambda + (x_n^k c(x) + \frac{\Delta g}{g})\bar{w}_\lambda = 0 \quad \text{in} \quad \Sigma_\lambda \setminus \{0\}. \tag{2.7}$$

Since $w_\lambda(\bar{x}) < 0$, we know $v_\lambda(\bar{x}) > v(\bar{x})$. From (2.4) we have: there exists $C > 0$, such that, as $|\lambda|$ is sufficiently large

$$v_\lambda(\bar{x}) = v(\bar{x}^\lambda) \leq \frac{C}{|\bar{x}^\lambda|^{n-2}}.$$

It follows that at point $\bar{x}$,

$$|x_n^k| c(\bar{x}) \leq \frac{n+2+2k}{n-2} \cdot (\frac{C}{|\bar{x}^\lambda|^{n-2}})^{\frac{2k+4}{n-2}} \cdot |\bar{x}|^k \leq C_1 |\bar{x}|^{-4-k}.$$

Noticing $\frac{\Delta g}{g}(\bar{x}) = -\frac{\alpha(n-2-\alpha)}{|\bar{x}|^2}$, we know that there exists $R_2 > R_1$ such that, as $\lambda < -R_2$, $\Delta g(\bar{x})/g(\bar{x}) + x_n^k c(\bar{x}) < 0$. Therefore, in view of the maximum principle, we know that $\bar{w}_\lambda$ can not attain an interior negative minimum in a neighborhood of $\bar{x}$. Contradiction!

Proposition 2.1 follows from the above claim directly.

Now we define

$$\lambda_0 = \sup\{\lambda < 0 \mid w_\mu(x) \geq 0 \ \text{ in } \tilde{\Sigma}_\mu \ \text{ for } \ \text{ all } \ -\infty < \mu < \lambda\}. \tag{2.8}$$

Proposition 2.2 *If $\lambda_0 < 0$, then $w_{\lambda_0} = 0$.*

Proof: We prove this proposition by contradiction. Suppose not, by the strong maximum principle we know that $w_{\lambda_0}(x) > 0$ in $\tilde{\Sigma}_{\lambda_0} \setminus T_{\lambda_0}$.

Claim: There exist some small constants: $r_0 \leq \min(|\lambda_0|/2, 1)$ and $\epsilon < 1$, such that

$$w_{\lambda_0}(x) \geq \frac{\epsilon}{2} \quad \text{in} \quad B_{r_0}(0) \setminus \{0\}.$$

Proof of the claim: Let $\varphi_2(x) = \frac{\epsilon}{2} - \frac{r^{n-2}\epsilon}{|x|^{n-2}} + \frac{|x|\epsilon}{2}$ in $B_{r_0}(0) \setminus B_r(0)$ for some small $r < r_0$, where ϵ and r_0 will be chosen. Since w_{λ_0} satisfies

$$\Delta w_{\lambda_0} + x_n^k c(x) w_{\lambda_0} = 0, \quad w_{\lambda_0} > 0 \quad \text{in} \quad B_{r_0}(0) \setminus \{0\},$$

we know

$$\Delta(w_{\lambda_0} - \varphi_2) = -x_n^k c(x) w_{\lambda_0} - \frac{\epsilon}{2} \cdot (n-1)|x|^{-1} \;\text{ in }\; B_{r_0}(0) \setminus B_r(0).$$

Set $B_{r_0}(0) \setminus B_r(0)= S_1 \cup S_1^c$ such that $S_1 = \{x \;:\; w_{\lambda_0} \geq 1 > \varphi_2\} \cap B_{r_0}(0) \setminus B_r(0)$. Obviously we only need to show that the claim holds in S_1^c.

In S_1^c we know $w_{\lambda_0} < 1$. Since $0 < 1/C \leq v_{\lambda_0}(x) \leq C$ for $x \in B_{r_0}(0)$ (here C is independent of r_0 whenever we choose $r_0 \leq |\lambda_0|/2)$, we have $v(x) \leq C+1$ in S_1^c. Thus, in S_1^c, there exists a constant $\bar{C} > 0$, such that

$$c(x) \leq \bar{C} < +\infty.$$

Now we fix r_0 small enough such that

$$(n-1)|x|^{-1} \geq 2|x_n^k| c(x) \qquad \forall x \in B_{r_0}(0) \setminus B_r(0), \tag{2.9}$$

and choose $\epsilon \leq \min\{\min_{\partial B_{r_0}(0)} w_{\lambda_0}, 1\}$. Set $S_1^c = S_{1,1} \cup S_{1,2}$ such that

$$S_{1,1} = \{x \;:\; -x_n^k c(x) w_{\lambda_0} \geq \frac{\epsilon}{2}(n-1)|x|^{-1}, \;\; x \in S_1^c\}.$$

In $S_{1,1}$, from (2.9), we know that $w_{\lambda_0} \geq \epsilon \geq \varphi_2$.

In $S_{1,2}$, we have $\Delta(w_{\lambda_0} - \varphi_2) \leq 0$. Also one can check that: On $\{x \;:\; -x_n^k c(x) w_{\lambda_0} = \frac{n-1}{2}|x|^{-1}\epsilon, \;\; x \in S^c\}$, as in $S_{1,1}$ we know $w_{\lambda_0} \geq \varphi_2$; On $\partial B_{r_0}(0)$, $w_{\lambda_0} - \varphi_2 \geq \epsilon - (\epsilon/2 + \epsilon/2) = 0$; On $\partial B_r(0)$, $w_{\lambda_0} - \varphi_2 \geq 0 - (\epsilon/2 - \epsilon + \epsilon/2) = 0$. That is on $\partial S_{1,2}$, $w_{\lambda_0} \geq \varphi_2$. Thus from the maximum principle, we know that $w_{\lambda_0} \geq \varphi_2$ in $S_{1,2}$.

It follows that $w_{\lambda_0} \geq \varphi_2$ in S_1^c. Let $r \to 0$, we have the claim.

Now we continue the proof of Proposition 2.2. By the definition of λ_0, there is a sequence $\lambda_l \to \lambda_0$ with $\lambda_l > \lambda_0$ such that $\inf_{\tilde{\Sigma}_{\lambda_l}} w_{\lambda_l} < 0$. As before, we consider $\bar{w}_{\lambda_l} = w_{\lambda_l}/g$ with $g(x) = |x|^{-\alpha}$. It follows from the above claim and $\bar{w}_{\lambda_l}(x) \to 0$ as $|x| \to \infty$ that there is P_l such that $\bar{w}_{\lambda_l}(P_l) = \min_{\tilde{\Sigma}_{\lambda_l}} \bar{w}_{\lambda_l}(x) < 0$. Similar discussion to the proof of Proposition 2.1, we also know that $P_l \in B_R(0)$ for some uniform constant R. Thus, as $l \to \infty$, $P_l \to \bar{x} \in T_{\lambda_0}$. Since $|\nabla \bar{w}_{\lambda_l}(P_l)| = 0$, we know $\partial \bar{w}_{\lambda_0}/\partial x_1(\bar{x}) = 0$. On the other hand, since $\bar{w}_{\lambda_0}$ satisfies (2.7) and $\bar{w}_{\lambda_0} > 0$ in $\tilde{\Sigma}_{\lambda_0} \setminus T_{\lambda_0}$, by Hopf Lemma we know $\frac{\partial \bar{w}_{\lambda_0}}{\partial x_1}(\bar{x}) > 0$. Contradiction! We complete the proof of Proposition 2.2.

If $\lambda_0 < 0$, from Proposition 2.2 we derive that $\lim_{x \to 0} v(x)$ exists, that is, $|x|^{n-2}u(x)$ tends to some constant $c_0 > 0$ as $x \to \infty$. If $\lambda_0 = 0$, we then begin to move the planes from positive x_1-axis to the origin and get either case 1: $v(x)$ is symmetry about origin on the x'-hyperplane (recall we can move the planes along any direction on x'-hyperplane), or case 2: $|x|^{n-2}u(x)$ tends to some constant $c_0 > 0$ as $x \to \infty$.

In the first case, from the property of the Kelvin transformation one easily gets that $u(x)$ is radial symmetry about the origin on the x'-hyperplane. Since we can choose the origin arbitrarily on the x'-hyperplane, we know that $u(x)$ is independent of x' and (1.2) becomes the following ODE:

$$u'' + x_n^k u^{\frac{n+2+2k}{n-2}} = 0, \quad u \geq 0, \quad \text{in } \mathbb{R}. \tag{2.10}$$

An elementary phase-plane argument shows that (2.10) has only trivial solution.

In the second case we know

$$\lim_{|x| \to \infty} |x|^{n-2} u(x) = c_0 > 0. \tag{2.11}$$

To complete the proof of Theorem 1.1, we only need to show that (1.2) have no positive solution under the condition (2.11), that is we only need to prove the following proposition.

Proposition 2.3 *There exists no positive solution of (1.2) which satisfies (2.11) if* $p > 1$.

Proof. We again prove this proposition by contradiction. Suppose that $u > 0$ solves (1.2) and satisfies (2.11) for some positive constant c_0.

Claim: $\frac{\partial u}{\partial x_n} \geq 0 \quad \forall x \in \mathbb{R}^n$.

Easy to see the claim contradicts to the fact $u(0) > 0$ and (2.11). Therefore, we only need to prove the claim under the contrary assumption (that is $u > 0$ solves (1.2) and satisfies (2.11)).

We use the method of moving planes again. This time we move planes along the positive x_n-direction.

For any $\lambda \in \mathbb{R}$, set

$$\begin{aligned}
&\Sigma_\lambda = \{x \mid \ x_n > \lambda\}, \ T_\lambda = \{x \mid \ x_n = \lambda\}, \\
&x^\lambda \text{ is the reflection of } x \text{ about } T_\lambda, \\
&u_\lambda(x) = u(x^\lambda), \ w_\lambda = u(x) - u_\lambda(x).
\end{aligned}$$

The claim can be proved through the following standard three steps. Here we outline the proof for completeness.

Step 1. There exists some constant $K > 0$ such that, if $\lambda < -K$, $w_\lambda \geq 0$ in Σ_λ.

The proof of this step is similar to that of Proposition 2.1. Here there is no singular point to worry about.

Then we can define

$$\lambda_0 = \sup\{\lambda \mid w_\mu(x) \geq 0 \text{ in } \Sigma_\mu \text{ for all } -\infty < \mu < \lambda\}.$$

Step 2. If $\lambda_0 \neq +\infty$, then $w_{\lambda_0} = 0$.

This can be proved as in Proposition 2.2. Again, there is no singular point to worry about.

Step 3. $\lambda_0 = +\infty$.

Proof. Assume $\lambda_0 < \infty$, from step 2 we know $w_{\lambda_0} = 0$. Therefore

$$\begin{array}{l} \Delta u + x_n^k u^p = 0 \quad \text{in } \mathbb{R}^n \\ \Delta u + (2\lambda_0 - x_n)^k u^p = 0 \quad \text{in } \mathbb{R}^n. \end{array}$$

It follows that $x_n = 2\lambda_0 - x_n$ for all $x_n \in \mathbb{R}$. Contradiction! We complete the proof of the claim.

3 Proofs of Theorem 1.3 and 1.4

Theorem 1.4 can be derived from Theorem 1.3 as the proof of Theorem 1.2, so we only give the proof of Theorem 1.3 in this section. Without loss of generality, we only consider $x_i = x_{n-1}$ in (1.3). We argue by contradiction.

Suppose $u \not\equiv 0$. Then, in view of the maximum principle and Hopf lemma, we know that $u > 0$ in $\overline{\mathbb{R}^n_+}$. Set

$$v(x) = \frac{1}{|x|^{n-2}} u(\frac{x}{|x|^2}),$$

then $v(x)$ satisfies

$$\left\{ \begin{array}{ll} \Delta v = 0, \quad v > 0, & \text{in } \mathbb{R}^n_+, \\ \frac{\partial v}{\partial x_n} = x_{n-1}^k v^{\frac{n+2k}{n-2}} & \text{on } \partial\mathbb{R}^n_+ \setminus \{0\}. \end{array} \right. \tag{3.12}$$

Again, we will move the planes which parallel to x_n-axis along x_1-direction.

In order to start to move planes, we still need the following lemma to take care of the possible singular point at the origin. We denote $B_r^+(0) := B_r(0) \cap \mathbb{R}^n_+$.

Lemma 3.1 *Assume that $v(x)$ satisfies*

$$\left\{ \begin{array}{ll} \Delta v(x) = 0 & \text{in } B^+_{1/2} \\ \frac{\partial v}{\partial x_n} = x_{n-1}^k v^{\frac{n+2k}{n-2}} & \text{on } \partial B^+_{1/2} \cap \partial\mathbb{R}^n_+ \setminus \{0\} \\ v > 0, \quad v \in C^2(\overline{B^+_{1/2}} \setminus \{0\}). & \end{array} \right.$$

If $v(x) \geq \epsilon$ on $\partial B_{1/2} \cap \mathbb{R}^n_+$ for some $\epsilon \leq 1$, then $v(x) \geq \epsilon/2$ in $\overline{B^+_{1/2}} \setminus \{0\}$.

Proof. Let $\varphi_3(x) = \frac{\epsilon}{2} - \frac{r^{n-2}\epsilon}{|x|^{n-2}} + \frac{x_n\epsilon}{2}$ in $B_{1/2}^+ \setminus B_r$ for some small r. Easy to see that $v - \varphi_3$ satisfies

$$\begin{cases} \Delta(v-\varphi_3) = 0 & \text{in} \quad B_{1/2}^+, \\ \frac{\partial(v-\varphi_3)}{\partial x_n} = x_{n-1}^k v^{\frac{n+2k}{n-2}} - \frac{\epsilon}{2} & \text{on} \quad \partial B_{1/2}^+ \cap \partial\mathbb{R}_+^n \setminus \{0\}, \\ v > 0, \qquad v \in C^2(\overline{B_{1/2}^+} \setminus \{0\}). \end{cases}$$

Notice $v - \varphi_3 > 0$ on $\partial B_{1/2} \cap \mathbb{R}_+^n$ and $\partial B_r \cap \mathbb{R}_+^n$. If there exists a point P_0 such that $v(P_0) - \varphi_3(P_0) = \min_{\overline{B_{1/2}^+} \setminus \overline{B_r^+}} (v(x) - \varphi_3(x)) < 0$, by the virtue of the maximum principle, we know that $P_0 \in \partial\mathbb{R}_+^n$, and thus $\partial(v-\varphi_3)/\partial x_n(P_0) \geq 0$. It follows that $v^{\frac{n+2k}{n-2}}(P_0) \geq \epsilon$, thus $v(P_0) > \varphi_3(P_0)$, contradiction! So $v - \varphi_3 > 0$ in $\overline{B_{1/2}^+} \setminus \overline{B_r^+}$ for some small r. Sending $r \to 0$, we complete the proof.

Now, we can start to move the planes.

For $\lambda < 0$ we define as in section 2 the following:

$$\begin{array}{l} \Sigma_\lambda = \{x \in \mathbb{R}_+^n \mid \ x_1 > \lambda\}, \ T_\lambda = \{x \in \mathbb{R}_+^n \mid \ x_1 = \lambda\}, \\ \tilde{\Sigma}_\lambda = \overline{\Sigma}_\lambda \setminus \{0\}, \ \ x^\lambda \ \text{ is the reflection of } x \text{ about } T_\lambda, \\ v_\lambda(x) = v(x^\lambda), \ w_\lambda = v(x) - v_\lambda(x). \end{array}$$

Then $w_\lambda(x)$ satisfies

$$\begin{cases} \Delta w_\lambda(x) = 0 & \text{in} \ \ \Sigma_\lambda, \\ \frac{\partial w_\lambda}{\partial x_n} = x_{n-1}^k c_1(x) w_\lambda & \text{on} \ \ \tilde{\Sigma}_\lambda \cap \partial\mathbb{R}_+^n \end{cases} \tag{3.13}$$

where $c_1(x) = \frac{n+2k}{n-2}\xi_1^{\frac{2k+2}{n-2}}(x)$, $\xi_1(x)$ is a positive function between v and v_λ.

Proposition 3.1 *There exists $L > 1$ such that, if $\lambda < -L$, $w_\lambda \geq 0$ in $\tilde{\Sigma}_\lambda$.*

Proof: As in [8], we choose an auxiliary function $g_1(x) = |z|^{-\alpha}$, where $0 < \alpha < n-2$, $z = x + (0, 0, ..., 1)$ and define $\bar{w}_\lambda = w_\lambda / g_1$. We only need to show $\bar{w}_\lambda \geq 0$ in $\tilde{\Sigma}_\lambda$ for λ negative enough.

Similar to the proof of Proposition 2.1, by using Lemma 3.1, we know that there exists a $L_1 > 0$ such that if $\lambda < -L_1$, $\bar{w}_\lambda \geq 0$ in $B_{1/2}^+ \setminus \{0\}$.

If for any $\lambda < -L_1$, $\inf_{\tilde{\Sigma}_\lambda} \bar{w}_\lambda < 0$, as in the proof of Proposition 2.1, we know that there exists a $\bar{x} \in \tilde{\Sigma}_\lambda$ such that $\bar{w}_\lambda(\bar{x}) = \inf_{\tilde{\Sigma}_\lambda} \bar{w}_\lambda < 0$. Direct computation shows that $\bar{w}_\lambda$ satisfies

$$\begin{cases} \Delta \bar{w}_\lambda + \frac{2}{g_1}\nabla g_1 \cdot \nabla \bar{w}_\lambda + \frac{\Delta g_1}{g_1}\bar{w}_\lambda = 0 & \text{in} \ \ \Sigma_\lambda \\ \frac{\partial \bar{w}_\lambda}{\partial x_n} = (x_{n-1}^k c_1(x) - \frac{1}{g_1} \cdot \frac{\partial g_1}{\partial x_n})\bar{w}_\lambda & \text{on} \ \tilde{\Sigma}_\lambda \cap \partial\mathbb{R}_+^n. \end{cases} \tag{3.14}$$

Since $\Delta g_1 / g_1 = -\alpha(n-2-\alpha)/|z|^2 < 0$, we know that $\bar{x} \in \partial\mathbb{R}_+^n$. Hence $\partial\bar{w}_\lambda/\partial x_n(\bar{x}) \geq 0$. From $w_\lambda(\bar{x}) < 0$, using a similar argument to the proof of Proposition 2.1, we have $|x_{n-1}^k| c_1(\bar{x}) \leq C|\bar{x}|^{-2-k}$. Also we know $-\frac{1}{g_1} \cdot \frac{\partial g_1}{\partial x_n} = \alpha/|z|^2$. It follows that if $\lambda < -L$ for some large $L > L_1$, $\partial\bar{w}_\lambda/\partial x_n(\bar{x}) < 0$. We derive a contradiction.

Proposition 3.2 *If* $\lambda_0 < 0$*, then* $w_{\lambda_0} = 0$.

Proof: We prove this proposition by contradiction. Suppose not, by the virtue of the maximum principle and Hopf lemma, we know $w_{\lambda_0}(x) > 0$ in $\tilde{\Sigma}_{\lambda_0} \setminus T_{\lambda_0}$.

Claim: There exist some small constants: $r_0 \le \min(|\lambda_0|/2, 1)$ and $\epsilon < 1$, such that

$$w_{\lambda_0}(x) \ge \frac{\epsilon}{2} \quad \text{in} \quad \overline{B^+_{r_0}(0)} \setminus \{0\}.$$

Proof of the claim: Let $\varphi_4 = \frac{\epsilon}{2} - \frac{r^{n-2}\epsilon}{|x|^{n-2}} + \frac{\epsilon x_n}{2}$ in $B^+_{r_0} \setminus B^+_r$ for some small $r < r_0$, where ϵ and r_0 will be chosen. Since w_{λ_0} satisfies

$$\begin{cases} \Delta w_{\lambda_0} = 0 \quad w_{\lambda_0} \ge 0 & \text{in} \quad B^+_{r_0}(0) \\ \frac{\partial w_{\lambda_0}}{\partial x_n} = x^k_{n-1} c_1(x) w_{\lambda_0} & \text{on} \quad \partial B^+_{r_0} \cap \partial \mathbb{R}^n_+ \setminus \{0\}, \end{cases}$$

we know

$$\begin{cases} \Delta(w_{\lambda_0} - \varphi_4) = 0 & \text{in} \quad B^+_{r_0}(0) \\ \frac{\partial (w_{\lambda_0} - \varphi_4)}{\partial x_n} = x^k_{n-1} c_1(x) w_{\lambda_0} - \frac{\epsilon}{2} & \text{on} \quad \partial B^+_{r_0} \cap \partial \mathbb{R}^n_+ \setminus \{0\}. \end{cases} \tag{3.15}$$

We want to show that for a suitable small r_0 and $\epsilon < \min\{\min_{\partial B_{r_0}} w_{\lambda_0}(x), 1\}$,

$$w_{\lambda_0}(x) \ge \varphi_4(x) \quad \forall x \in \overline{B^+_{r_0}} \setminus \overline{B^+_r}. \tag{3.16}$$

If not, due to (3.15) and the fact that $w_{\lambda_0} - \varphi_4 \ge 0$ on $(\partial B_{r_0} \cup \partial B_r) \cap \mathbb{R}^n_+$, we know that there exists a $P_0 \in \partial \mathbb{R}^n_+$ such that $w_{\lambda_0}(P_0) - \varphi_4(P_0) = \inf_{\tilde{\Sigma}_{\lambda_0}} (w_{\lambda_0} - \varphi_4) < 0$ and

$$\frac{\partial (w_{\lambda_0} - \varphi_4)}{\partial x_n}(P_0) \ge 0.$$

Notice $w_{\lambda_0}(P_0) \le \varphi_4(P_0) < \epsilon$, as in the proof of Proposition 2.2, it yields that $c_1(P_0) \le C$ (C is independent of r_0 whenever we choose $r_0 \le |\lambda_0|/2$). Now we choose r_0 small enough, such that

$$r_0 C < \frac{1}{2}.$$

Then, we have

$$\frac{\partial (w_{\lambda_0} - \varphi_4)}{\partial x_n}(P_0) < 0.$$

Contradiction! Thus, we have shown that (3.16) holds for some suitable chosen r_0 and ϵ. Sending $r \to 0$, we complete the proof of the claim.

Now we continue the proof of Proposition 3.2. By the definition of λ_0, there exists a sequence $\lambda_l \to \lambda_0$ with $\lambda_l > \lambda_0$ such that $\inf_{\tilde{\Sigma}_{\lambda_l}} w_{\lambda_l} < 0$. As before, we consider $\bar{w}_{\lambda_l} = w_{\lambda_l}/g_1$ with $g_1(x)$ defined in the proof of Proposition 3.1.

It follows from the above claim, $\bar{w}_{\lambda_l}(x) \to 0$ as $|x| \to \infty$ and $\Delta g_1/g_1 < 0$ in (3.14) that there exists $P_l \in \partial\mathbb{R}^n_+$ such that $\bar{w}_{\lambda_l}(P_l) = \inf_{\tilde{\Sigma}_{\lambda_l}} \bar{w}_{\lambda_l}$. Therefore $\partial\bar{w}_{\lambda_l}/\partial x_1(P_l) = 0$. Similar discussion to the proof of Proposition 3.1 we can conclude that $P_l \in B_R(0)$ for some large constant R uniformly. Thus as $l \to \infty$, $P_l \to \bar{x} = T_{\lambda_0} \cap \partial\mathbb{R}^n_+$, and $\partial\bar{w}_{\lambda_0}/\partial x_1(\bar{x}) = 0$.

In order to derive a contradiction, we still need the following technical lemma to take care of the corner point $\bar{x}$. Without loss of generality, we assume $\lambda_0 = -1$ and $\bar{x} = (-1, 0, ..., 0)$. The proof is almost the same as that of Lemma 2.4 in [8], we include it here for completeness.

Lemma 3.2 *Assume that w_{λ_0} satisfies (3.13), $w_{\lambda_0} > 0$ in $\tilde{\Sigma}_{\lambda_0}$ for some $\lambda_0 < 0$ and $\bar{x} \in T_{\lambda_0} \cap \partial\mathbb{R}^n_+$, then*

$$\frac{\partial w_{\lambda_0}}{\partial x_1}(\bar{x}) > 0. \tag{3.17}$$

Proof. Set $\Omega = \{x \ : \ x \in B_1^+ \setminus \bar{B}^+_{1/2}, \quad x_n < 1/4\}$ and for some $\alpha > \max\{n/2, n-3\}$,

$$h(x) = \beta(|x'|^{-\alpha} - 1)(x_n + \mu), \quad \varphi_5(x) = h(x) - \frac{1}{|x|^{n-2}} h(\frac{x}{|x|^2}), \quad x \in \Omega,$$

where $0 < \beta, \mu < 1$ will be chosen later. A direct computation yields

$$\Delta\varphi_5 \geq 0.$$

Consider $B(x) = w_{\lambda_0} - \varphi_5$; it follows that

$$\begin{cases} \Delta B \leq 0, & \text{in } \ \Omega, \\ \frac{\partial B}{\partial x_n} = x^k_{n-1} c_1 w_{\lambda_0} - \frac{\partial \varphi_5}{\partial x_n} & \text{on } \ \partial\Omega \cap \partial\mathbb{R}^n_+. \end{cases} \tag{3.18}$$

For some suitable chosen β and μ, we want to show

$$B(x) \geq 0, \quad \forall x \in \Omega. \tag{3.19}$$

Using $w_{\lambda_0} > 0$ in $\Sigma_{\lambda_0} \setminus \{0\}$, we can choose $\beta_0 > 0$ such that for all $0 < \beta < \beta_0$, $B(x) \geq 0$ on $\partial\Omega \cap \{\partial B_{1/2} \cup \{x_n = 1/4\}\}$. Also, one can see that $B(x) \geq 0$ on $\partial\Omega \cap \partial B_1$. Suppose the contrary of (3.19), there exists some $\tilde{x} = (\tilde{x}', d) \in \bar{\Omega}$ such that

$$B(\tilde{x}) = \min_{\bar{\Omega}} B(x) < 0.$$

It follows that $d = 0$,

$$w_{\lambda_0}(\tilde{x}) < \beta\mu((|\tilde{x}'|^{-\alpha} - 1)(|\tilde{x}'|^{-n+\alpha+2} + 1) \tag{3.20}$$

and

$$\frac{\partial B}{\partial x_n}(\tilde{x}) \geq 0.$$

A Simple calculation yields

$$\frac{\partial \varphi_5}{\partial x_n}(\tilde{x}) = \beta(|\tilde{x}'|^{-\alpha} - 1)(|\tilde{x}'|^{-n+\alpha} + 1).$$

Combining the above two inequalities we have

$$\tilde{x}_{n-1}^k c_1(\tilde{x}) w_{\lambda_0}(\tilde{x}) - \beta(|\tilde{x}'|^{-\alpha} - 1)(|\tilde{x}'|^{-n+\alpha} + 1) \geq 0. \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$\tilde{x}_{n-1}^k c_1(\tilde{x})\mu > 1.$$

If we choose $0 < \mu < \min_{1/2 \leq |x| \leq 1}(\tilde{x}_{n-1}^k c_1(x) + 1)^{-1}$ from the beginning, we reach a contradiction, thus (3.19) holds.

Notice $B(\bar{x}) = 0$, we have

$$\frac{\partial B}{\partial x_1}(\bar{x}) \geq 0.$$

It follows that

$$\frac{\partial w_{\lambda_0}}{\partial x_1}(\bar{x}) = \frac{\partial B}{\partial x_1}(\bar{x}) + \frac{\partial \varphi_5}{\partial x_1}(\bar{x}) \geq \frac{\partial \varphi_5}{\partial x_1}(\bar{x}) = 2\alpha\beta\mu > 0.$$

We complete the proof of Lemma 3.2, therefore complete the proof of Proposition 3.2.

Now, as in Section 2, we have two case. Case 1: $\lambda_0 < 0$, we know

$$\lim_{|x| \to \infty} |x|^{n-2} u(x) = c_0 > 0. \tag{3.22}$$

Case 2: $v(x)$ is radial symmetry about the origin on the x''-hyperplane (we write $x = (x'', x_{n-1}, x_n)$). Due to the property of Kelvin transformation, we know that $u(x)$ is radial symmetry about the origin on the x''-hyperplane. Since we can choose the origin arbitrarily on the x''-hyperplane, we conclude that $u(x)$ just depends on x_n and x_{n-1} in this case.

Proposition 3.3 *There exists no positive solution of (1.3) which satisfies (3.22) if* $p > 1$.

This proposition yields that case 1 will not happen. Therefore we complete the proof of Theorem 1.3 by completing the proof of the above proposition.

Proof of Proposition 3.3. The proof is the same as that of Proposition 2.3. Assume $u > 0$, due to (3.22), we can apply moving planes directly along the x_{n-1}-direction, and get

$$\frac{\partial u}{\partial x_{n-1}} \geq 0.$$

Then the contradiction comes from the above, $u(0) > c > 0$ and (3.22). We leave these details to readers.

Remark. Our method heavily depends on the invariance of the equation under the Kelvin transformation, therefore, we can only classify some equations with discrete exponents and can **not** prove our Theorems for all p less than or equal to the critical exponents. The natural question is: Does Theorem 1.1 still hold for any $1 < p \leq (n+2)/(n-2)$?

Also it might be interesting to seek that Theorem 1.3 holds for some continuous range of q.

4 Application

The Liouville theorems we derived here are mainly applied to get some *a priori* estimates in the study of certain elliptic boundary value problems. As a consequence, one can obtain some existence results via blowup argument and degree theory, see for instance [6] or [1]. Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded smooth domain, we here present an existence result concerning the same equations which was discussed in [1]:

$$\begin{cases} \mathcal{L}u + a(x)g(u) = 0, \quad u > 0, & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega \end{cases} \tag{4.23}$$

where, $\mathcal{L}$ is an uniformly elliptic linear operator:

$$\mathcal{L} = a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + b_i(x)\frac{\partial}{\partial x_i} + c(x),$$

with $a_{ij}(x) \in C^2(\overline{\Omega})$, $b_i(x) \in C^1(\overline{\Omega})$, $c(x) \in L^\infty$ and

$$c_0|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq C_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \forall x \in \Omega,$$

for some $c_0,\ C_0 > 0$. The boundary operator B is one of the following

$$Bu := u \tag{4.24}$$

$$Bu := \nu_j a_{jk} u_{x_k} + \alpha(x)u, \tag{4.25}$$

where $\nu = (\nu_1, ..., \nu_n)$ denotes the exterior unit normal on $\partial\Omega$, α is a given continuous nonnegative function on $\partial\Omega$.

We assume that g is a C^1 function on $\mathbb{R}^+$ with

$$g(0) = g'(0) = 0 \quad \text{and} \quad \lim_{s \to \infty} g(s)/s^{\frac{n+2}{n-2}} = l > 0. \tag{4.26}$$

The function $a(x)$ which we are considering here is C^2 and changes sign, that is, both

$$\Omega^+ := \{x \in \Omega \quad : \quad a(x) > 0\} \quad \text{and} \quad \Omega^- := \{x \in \Omega \quad : \quad a(x) < 0\}$$

are nonempty. We assume that

$$\Gamma := \overline{\Omega^+} \cap \overline{\Omega^-} \subset \Omega \quad \text{and} \quad \nabla a(x) \neq 0 \ \ \forall x \in \Gamma. \tag{4.27}$$

Let $\lambda_1(-\mathcal{L})$ be the principle eigenvalue of the operator $-\mathcal{L}$ in Ω under the boundary condition $Bu = 0$. Then the existence result can be stated as:

Theorem 4.1 *Assume (4.26), (4.27), $\lambda_1(-\mathcal{L}) > 0$ and the dimension $n \geq 3$ is an even number. Then problem (4.23) has one solution.*

With the aid of Theorem 1.2, we can prove the above theorem exactly in the same way as in [1]. We omit details here.

Remark 4.1 *Chen and Li also obtained some existence results concerning the prescribing scalar curvature problem in [4]. However, their method can not be applied to general equation (4.23).*

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