Parabolic equations related to curve motion

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We study curve motions based on differential equations. Curve motion equations are classified into two types: adaptive equations (which depend on the choice of coordinate systems) and non-adaptive equations. Examples from both types of equations are studied, and the global existence for these equations is proved based on integral estimates.

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1. Introduction

Let $\mathbf{X}(\mathbf{u}) : [0, 2\pi] \to \mathbb{R}^2$ be a closed plane curve embedded in $\mathbb{R}^2$. The evolution of $\mathbf{X}(\mathbf{u})$ along its normal direction in $\mathbb{R}^2$:

$$\mathbf{X}_t = \phi(k)\mathbf{N}, \qquad (1.1)$$

where and throughout the paper $\mathbf{N}$ is the inner unit norm of the curve $\mathbf{X}$, k is its curvature and $\phi(\cdot)$ is a given function, is widely studied since the early work of Gage [6] and Gage and Hamilton [7]. For $\phi(x) = x$, it is the well-known *curve shortening flow*. For $\phi(x) = x^{1/3}$, it is equivalent to the *affine curvature flow*, which was studied by Sapiro and Tannenbaum [12], and by Alvarez et al. [2], see also, the work of Ni and Zhu [9,10]. Most of the studies in curve motion problems have direct impacts in digital image processing, thanks to the powerful level set methods of Osher and Sethian [11].

In this paper we shall study curve motion equations like (1.1) from viewpoint of gradient flow of certain total energy. Our study is motivated by our early work on conformal curvature flow [9,10]. For any positive, 2π-periodic function $\rho(\theta) \in C^2[0, 2\pi]$, and a given positive parameter α, in [9] we define its *α-flow constant* by

$$R_\rho^\alpha = \rho^3(\alpha\rho_{\theta\theta} + \rho). \qquad (1.2)$$

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R_ρ^α is introduced in [9] as α-scalar curvature if ρ is given via a conformal transform, which generalizes the notion of scalar curvature as well as the affine curvature for one-dimensional curves. For instance, if $\alpha = 1$ and ρ is one-third power of the curvature of a given curve, then the 1-scalar curvature is in fact the affine curvature of the curve. The corresponding curvature flows were introduced in [9]. The global existence and the convergence of curvatures for these flows were obtained in [10]. The parametric curve $\mathbf{X}(\theta, \mathbf{t}) : [0, 2\pi] \to \mathbb{R}^2$ can be recovered from $\rho(\theta, t)$ by

$$\mathbf{X}(\theta, \mathbf{t}) := \left(\int\limits_0^\theta \frac{\cos \alpha}{\rho^3} \, d\alpha, \int\limits_0^\theta \frac{\sin \alpha}{\rho^3} \, d\alpha \right).$$

Here we shall consider (1.2) from pure differential equation point of view. Define the average α-*flow constant* by

$$\overline{R}_\rho^\alpha = \frac{\int_0^{2\pi} \rho(\alpha \rho_{\theta\theta} + \rho) \, d\theta}{\int_0^{2\pi} \rho^{-2} \, d\theta}. \tag{1.3}$$

We introduce our motion equation as

$$\rho_t = \frac{1}{4} \big(R_\rho^\alpha - \overline{R}_\rho^\alpha \big) \rho, \quad \text{that is} \quad \rho_t = \frac{\alpha}{4} \rho^4 \rho_{\theta\theta} + \frac{1}{4} \rho^5 - \frac{1}{4} \overline{R}_\rho^\alpha \rho. \tag{1.4}$$

We will show

Theorem 1. *For $\alpha \geqslant 4$, if $\rho(\theta, t)$ satisfies (1.4) with $\rho(\theta, 0) = \rho_0(\theta)$, where $\rho_0(\theta) \in C^1[0, 2\pi]$ is a positive, 2π-periodic function, then $\rho(\theta, t)$ exists for all $t > 0$.*

The global existence and convergence of flow (1.4) for $\alpha = 1$ are known if ρ satisfies the orthogonal conditions (so that the flow is equivalent to an affine flow, see, for example, [12] and [10]):

$$\int\limits_0^{2\pi} \frac{\cos \alpha}{\rho_0^3} \, d\alpha = \int\limits_0^{2\pi} \frac{\sin \alpha}{\rho_0^3} \, d\alpha = 0. \tag{1.5}$$

The global existence and convergence of flow (1.4) for $\alpha = 4$ are also known since $\psi(\theta, t) = \rho(\theta/2, t)$ satisfies (1.4) with $\alpha = 1$ and $\psi_0(\theta) = \rho_0(\theta/2)$ satisfies orthogonal condition (1.5). The global existence and convergence of flow (1.4) for $\alpha \in (0, 4)$ are open.

Comparing with curvature flow equations studied in [9,10], we classify flow (1.4) as *adaptive flows* since the 2π-periodic function ρ can be viewed as the polar distance function of a curve, and the deformation of such curve depends on not only the shape of the figures but also the location (or coordinate systems). On the other hand, a flow which does not depend on the choice of coordinate system, such as conformal curvature flow, is classified as a *non-adaptive flow*. More generally, if ρ is given as a curvature function of a given curve in (1.4), then it is a non-adaptive flow. Unfortunately, for parameter $\alpha \neq 1, 4$ and ρ given as a function of curvature, a closed curve may not be closed anymore under the flow (1.4).

Another family of non-adaptive flows is the following curvature flows:

$$k_t = k^2 \cdot (\tau_{\theta\theta} + \tau) \tag{1.6}$$

where $k(\theta, t)$ is the curvature and τ is a function of k. Under the flow, one can check that the orthogonal condition

$$\int\limits_0^{2\pi} \frac{\cos\theta}{k}\, d\theta = \int\limits_0^{2\pi} \frac{\sin\theta}{k}\, d\theta = 0 \tag{1.7}$$

holds for all $t > 0$, which guarantees that $k(\theta, t)$ is the curvature function of a closed curve. In fact, (1.6) is equivalent to the generalized curve shortening flow (1.2) with $\phi(k) = \tau$. From PDE point of view, we shall obtain a slight general result:

Theorem 2. *Assume that $\tau = k^p + \lambda$ for $p > 1$ and $\lambda \geqslant 0$ in (1.6). Then solution $k(\theta, t)$ to (1.6) with positive $k(\theta, 0) = k_0(\theta) \in C^1(S^1)$ exists for all $t > 0$.*

If initial condition $k_0(\theta)$ satisfies (1.7), Theorem 2 yields another proof for the global existence of general curve flow equation due to Andrews [4]. One can see from the isoperimetric ration that a closed curve under the normalized flow when k is the curvature function will converge to a circle. The general result in Theorem 2 is new since $k(\theta, t)$ may not satisfy condition (1.7). It is an interesting question to ask whether the result holds for $p = 1$ in Theorem 2. We suspect that the solution may blow up in finite time without assumption of the orthogonal condition, see Remark 3 for more details.

The approach to prove Theorem 2 appeared in the early work [10]. Moreover, such an approach enables us to obtain the global existence to a more general adaptive flow. For any positive, 2π-periodic function $\rho(\theta) \in C^2[0, 2\pi]$, and given positive parameters α and p, we define its *α-shorten-flow constant* by

$$R_p^\alpha = \rho\big(\alpha(\rho^p)_{\theta\theta} + \rho^p\big). \tag{1.8}$$

The average *α-shorten-flow constant* is given by

$$\overline{R}_p^\alpha = \frac{\int_0^{2\pi} R_p^\alpha \cdot \rho^{p-1}\, d\theta}{\int_0^{2\pi} \rho^{p-1}\, d\theta}$$

for $p \neq 1$, and

$$\overline{R}_1^\alpha = \frac{1}{2\pi} \int\limits_0^{2\pi} R_1^\alpha\, d\theta$$

for $p = 1$.

Consider the normalized flow

$$\rho_t = \big(R_p^\alpha - \overline{R}_p^\alpha\big)\rho. \tag{1.9}$$

We will show:

Theorem 3. *Assume that $p > 1$ and $\alpha > 0$ in (1.9). Then for any positive function $\rho_0(\theta) \in C^1(S^1)$, solution $\rho(\theta, t)$ satisfying (1.9) with $\rho(\theta, 0) = \rho_0(\theta)$ exists for all $t > 0$.*

If $\alpha = k^2$ for certain natural number k, Theorem 3 is a special case of Theorem 2. In fact, the proof of Theorem 3 is quite similar to that of Theorem 2. The method seems not work for $p \leqslant 1$. The case of $p \in [1/3, 1]$ can be settled by utilizing the proof of Theorem 1. We have

Theorem 4. *Assume that $p \in [1/3, 1]$ and $\alpha \geqslant 4$ in (1.9). Then for any positive function $\rho_0(\theta) \in C^1(S^1)$, solution $\rho(\theta, t)$ satisfying (1.9) with $\rho(\theta, 0) = \rho_0(\theta)$ exists for all $t > 0$.*

Remark 1. In fact, for $p = 1/3$, Eq. (1.9) is equivalent to Eq. (1.4).

Again, the above result for $\alpha = 4$ and $p = 1/3$ is known since $\psi(\theta, t) = \rho(\theta/2, t)$ satisfies (1.4) with $\alpha = 1$ and $\psi_0(\theta) = \rho_0(\theta/2)$ satisfies orthogonal condition (1.5) (see, for example, [4] and [5]).

The adaptive flows corresponding to conformal curvature flow will be studied in Section 2 and Theorem 1 is proved in Section 2. The curve shorten flows will be re-visited, and the global existence for general non-homogeneous flow (Theorem 2) is proved in Section 3. Finally we will study the adaptive flows corresponding to curve shortening flows in Section 4. The proof of Theorem 3 for $p > 1$ and that of Theorem 4 for $p \leqslant 1$ are completely different. All of them are presented in the last section. The C^1 assumption on initial value may be weakened to L^∞ assumption via standard regularity argument. Details are left for interested readers.

2. Adaptive flows

In this section, we shall discuss adaptive flows, motivated by our early study of conformal curvature flows in [9,10]. Before we derive estimates on function ρ which satisfies (1.4), we establish certain general properties for the flow for $\alpha > 0$.

2.1. Basic properties

In this subsection, we assume that $\rho > 0$. This assumption is satisfied by solutions to the flow with positive initial data.

It is easy to see from the definition of the flow that along flow (1.4),

$$\partial_t \int_0^{2\pi} \rho^{-2} \, d\theta = \frac{1}{2} \int_0^{2\pi} \left(\bar{R}_\rho^\alpha - R_\rho^\alpha \right) \rho^{-2} \, d\theta = 0. \tag{2.1}$$

Due to this, we can assume, through the proof of Theorem 1, that $\int_0^{2\pi} \rho^{-2} \, d\theta = 2\pi$.

Lemma 1. *Along flow* (1.4), $R := R_\rho^\alpha$ *and* $\bar{R} := \bar{R}_\rho^\alpha$ *satisfy*

$$R_t = \frac{\alpha}{4} \rho^2 \left(\rho^2 R_\theta \right)_\theta + R(R - \bar{R}) \tag{2.2}$$

and

$$\partial_t \bar{R} = \frac{1}{4\pi} \int_0^{2\pi} (R - \bar{R})^2 \rho^{-2} \, d\theta. \tag{2.3}$$

Proof. The proof is essentially given in [9] in the language of conformal geometry. We mimic the computation here:

$$R_t = \left(\rho^3 (\alpha \rho_{\theta\theta} + \rho) \right)_t = 3\rho^2 \rho_t (\alpha \rho_{\theta\theta} + \rho) + \rho^3 (\alpha \rho_{t\theta\theta} + \rho_t)$$

$$= 3\rho^{-1} \rho_t R + \rho^3 \left(\alpha \left(\rho \cdot \frac{\rho_t}{\rho} \right)_{\theta\theta} + \rho \cdot \frac{\rho_t}{\rho} \right)$$

$$= \frac{3}{4} (R - \bar{R}) R + \alpha \rho^2 \left(\rho^2 \cdot \left(\frac{\rho_t}{\rho} \right)_\theta \right)_\theta + R \cdot \frac{\rho_t}{\rho}$$

$$= \frac{\alpha}{4} \rho^2 \left(\rho^2 R_\theta \right)_\theta + R(R - \bar{R}).$$

Here, the magic computation

$$\rho^3 \left(\alpha \left(\rho \cdot \frac{\rho_t}{\rho} \right)_{\theta\theta} + \rho \cdot \frac{\rho_t}{\rho} \right) = \alpha \rho^2 \left(\rho^2 \cdot \left(\frac{\rho_t}{\rho} \right)_\theta \right)_\theta + R \cdot \frac{\rho_t}{\rho}$$

in fact is due to certain conformal covariant property, discovered in our early work [9] (see the proof of Proposition 1 there).

Since

$$\int_0^{2\pi} \rho^{-2} \, d\theta = 2\pi \quad \text{and} \quad \overline{R} = \frac{1}{2\pi} \int_0^{2\pi} R \rho^{-2} \, d\theta,$$

we have

$$\partial_t \overline{R} = \frac{1}{2\pi} \int_0^{2\pi} R_t \rho^{-2} \, d\theta - \frac{1}{\pi} \int_0^{2\pi} R \rho^{-3} \rho_t \, d\theta$$

$$= \frac{\alpha}{8\pi} \int_0^{2\pi} (\rho^2 R_\theta)_\theta \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} R(R - \overline{R}) \rho^{-2} \, d\theta - \frac{1}{4\pi} \int_0^{2\pi} R(R - \overline{R}) \rho^{-2} \, d\theta$$

$$= \frac{1}{4\pi} \int_0^{2\pi} R(R - \overline{R}) \rho^{-2} \, d\theta$$

$$= \frac{1}{4\pi} \int_0^{2\pi} (R - \overline{R})^2 \rho^{-2} \, d\theta. \quad \square$$

Remark 2. In fact, one can directly derive (2.3) from flow equation (1.4) without using (2.2). See for example the proof of Proposition 2 below.

From Lemma 1, it is clear that along flow (1.4), $\overline{R}^\alpha_\rho$ is monotonically increasing. In order to prove the global existence of the flow for certain α, we need to show that $\overline{R}^\alpha_\rho$ is bounded above for such α. In fact, for $\alpha \geqslant 4$, such upper bound follows from a sharp Sobolev type inequality on S^1, due to Ni and Zhu, and Hang independently (see, e.g., Theorem 2 in [9], and Hang [8]).

Lemma 2. *(See [9,8].) For any positive $u(\theta) \in H^1(S^1)$,*

$$\int_0^{2\pi} \left(u_\theta^2 - \frac{1}{4} u^2 \right) d\theta \cdot \int_0^{2\pi} u^{-2}(\theta) \, d\theta \geqslant -\pi^2,$$

and the equality holds if and only if

$$u(\theta) = c \sqrt{ \lambda^2 \cos^2 \frac{\theta - \alpha}{2} + \lambda^{-2} \sin^2 \frac{\theta - \alpha}{2} }$$

for some $\lambda, c > 0$ and $\alpha \in [0, 2\pi)$.

Corollary 1. *If $\alpha > 4$, then for all $u(\theta) \in H^1(S^1)$ and $u > 0$,*

$$\int_0^{2\pi} (\alpha u_\theta^2 - u^2)\, d\theta \cdot \int_0^{2\pi} u^{-2}(\theta)\, d\theta \geqslant -4\pi^2,$$

and the equality holds if and only if $u(\theta) = constant$.

Proof.

$$\int_0^{2\pi} (\alpha u_\theta^2 - u^2)\, d\theta \cdot \int_0^{2\pi} u^{-2}(\theta)\, d\theta \geqslant -4\pi^2 + \int_0^{2\pi} (\alpha - 4)u_\theta^2\, d\theta \cdot \int_0^{2\pi} u^{-2}(\theta)\, d\theta$$

$$\geqslant -4\pi^2,$$

and the equality holds if and only if $\int_0^{2\pi} u_\theta^2 = 0$, that is, $u(\theta) = constant$. □

From the definition of *α-flow constant*, we obtain immediately

Corollary 2.

(I) *If $\alpha > 4$, and $\int_0^{2\pi} \rho^{-2}\, d\theta = 2\pi$, then*

$$\overline{R}_\rho^\alpha \leqslant 1,$$

and the equality holds if and only if $\rho(\theta) = constant$.

(II) *If $\alpha = 4$, and $\int_0^{2\pi} \rho^{-2}\, d\theta = 2\pi$, then*

$$\overline{R}_\rho^\alpha \leqslant 1,$$

and the equality holds if and only if

$$\rho(\theta) = c\sqrt{\lambda^2 \cos^2 \frac{\theta - \alpha}{2} + \lambda^{-2} \sin^2 \frac{\theta - \alpha}{2}}$$

for some $\lambda, c > 0$ and $\alpha \in [0, 2\pi)$.

2.2. Global existence for $\alpha \geqslant 4$

We shall prove Theorem 1 in this subsection. Throughout this subsection we always assume that $\alpha \geqslant 4$.

Suppose that $\rho(\theta, t)$ satisfies (1.4) and $\rho(\theta, 0) = \rho_0(\theta)$, where $\rho_0(\theta)$ is a positive, 2π-periodic function in $C^0(S^1)$. Then the local existence follows from the standard argument via fixed point theorem. The global existence follows from parabolic estimates and the following *a priori* estimate.

Proposition 1. *Suppose $\rho(\theta, t)$ satisfies (1.4) and $\rho(\theta, 0) = \rho_0(\theta) \in C^1(S^1)$. If $\rho_0(\theta) > 0$ and $\alpha \geqslant 4$, then for any given $t_0 > 0$, there is a positive constant $c = c(t_0) > 0$ such that*

$$\frac{1}{c(t_0)} \leqslant \rho(\theta, t) \leqslant c(t_0), \quad \forall t \in [0, t_0].$$

Proof. The essential idea is the same as that in [10]. For simplicity, we use R to replace R_ρ^α and $\bar{R}$ to replace $\bar{R}_\rho^\alpha$ in the proof. From (2.2) we know that

$$R_t + \bar{R}R \geqslant \frac{\alpha}{4}\rho^2\big(\rho^2 R_\theta\big)_\theta.$$

It follows from the Maximum Principle that

$$R(\theta, t) \geqslant \min_\theta R(\theta, 0) \cdot e^{-\int_0^t \bar{R}\, d\tau}. \tag{2.4}$$

From Corollary 2 we know that $\bar{R} \leqslant 1$. Thus there is a constant $c_1(\rho_0(\theta))$ (it might be negative), depending on $\rho_0(\theta)$, such that

$$R(\theta, t) \geqslant c_1\big(\rho_0(\theta)\big), \quad t \in [0, t_0]. \tag{2.5}$$

It then follows from (1.4) that there is a positive constant $c_2(\rho_0(\theta), t_0)$, depending on $\rho_0(\theta)$ and t_0, such that

$$\rho(\theta, t) = \rho_0(\theta) \cdot e^{\frac{1}{4}\int_0^t (R-\bar{R})\, d\tau} \geqslant c_2\big(\rho_0(\theta), t_0\big) > 0, \quad t \in [0, t_0]. \tag{2.6}$$

To estimate the upper bound on $\rho(\theta, t)$, we first observe that for fixed t, ρ satisfies

$$\alpha \rho_{\theta\theta} + \rho = R\rho^{-3}, \quad \rho > 0, \quad \text{and} \quad \int_0^{2\pi} \rho^{-2}\, d\theta = 2\pi.$$

Multiplying the above by ρ and then integrating it from 0 to 2π, we obtain

$$\int_0^{2\pi} \rho^2\, d\theta - \alpha \int_0^{2\pi} \rho_\theta^2\, d\theta = \int_0^{2\pi} R\rho^{-2}\, d\theta = 2\pi \bar{R} \geqslant 2\pi \bar{R}_0, \tag{2.7}$$

where $\bar{R}_0 = \bar{R}(t)|_{t=0}$. The last inequality follows from (2.3) in Lemma 1. Let $M(t) = |\{\theta:\ \rho(\theta, t) \geqslant 2\}|$. Then (2.6) implies

$$2\pi = \int_0^{2\pi} \rho^{-2}\, d\theta = \int_{\rho \geqslant 2} \rho^{-2}\, d\theta + \int_{\rho < 2} \rho^{-2}\, d\theta$$

$$\leqslant \frac{M(t)}{4} + \big(2\pi - M(t)\big)c_2\big(\rho_0(\theta), t_0\big)^{-2}.$$

Therefore there exists $\delta(t_0) > 0$, such that $2\pi - M(t) \geqslant \delta(t_0)$. That is

$$\big|\{\theta:\ \rho(\theta, t) \leqslant 2\}\big| > \delta(t_0), \quad \text{for } t \in [0, t_0].$$

If $\sup_{t\in[0,t_0)} \int_0^{2\pi} \rho^2(t)\, d\theta = \infty$, then there exists a sequence $t_i \to t_* \leqslant t_0$, such that $\int_0^{2\pi} \rho^2(t_i)\, d\theta = \tau_i^2 \to \infty$ as $i \to \infty$. We define $v_i(\theta) = \rho(\theta, t_i)/|\tau_i|$. It follows from (2.7) that v_i satisfies

$$\int\limits_0^{2\pi} v_i^2 \, d\theta = 1, \quad \text{and} \quad \alpha \int\limits_0^{2\pi} (v_i)_\theta^2 \, d\theta \leqslant \int\limits_0^{2\pi} v_i^2 \, d\theta - \frac{2\pi \bar{R}_0}{\tau_i^2} \leqslant c_3,$$

which yields that $\{v_i\}$ is a bounded set in $H^1 \hookrightarrow C^{0,\frac{1}{2}}$. Therefore up to a subsequence $v_i \rightharpoonup v_0$ in H^1 weakly and $v_0 \in C^{0,\frac{1}{2}}$. From Sobolev compact embedding we know that v_0 satisfies $\int_0^{2\pi} v_0^2 \, d\theta = 1$, $v_0(\theta) \geqslant 0$, $|\{\theta: v_0(\theta) = 0\}| \geqslant \delta_1(t_0) > 0$. Also, from the weak convergence $v_i \rightharpoonup v_0$ in H^1, we know that $\int_0^{2\pi} (v_0)_\theta^2 \, d\theta \leqslant \underline{\lim}_{i \to \infty} \int_0^{2\pi} (v_i)_\theta^2 \, d\theta$, thus

$$0 \leqslant \int\limits_0^{2\pi} v_0^2 \, d\theta - \alpha \int\limits_0^{2\pi} (v_0)_\theta^2 \, d\theta.$$

On the other hand, for an interval $I \subset (0, 2\pi)$ with positive measure if $u \in H^1(S^1)$ with $u = 0$ in I and $u \geqslant 0$, then for any $\alpha \geqslant 4$,

$$\int\limits_0^{2\pi} u^2 \, d\theta - \alpha \int\limits_0^{2\pi} (u)_\theta^2 \, d\theta \leqslant 0$$

and "$=$" holds if and only if $u \equiv 0$. Contradiction. Therefore $\int \rho^2 \, d\theta$ is bounded on $[0, t_0]$, so is $\int \rho_\theta^2 \, d\theta$. Thus $\rho(\theta, t)$ is bounded in $H^1 \hookrightarrow C^{0,\frac{1}{2}}$, which implies that there exists a $c(t_0) > 0$ such that

$$\frac{1}{c(t_0)} \leqslant \rho(\theta, t) \leqslant c(t_0), \quad t \in [0, t_0]. \qquad \square$$

2.3. Limiting shapes

It is clear from the proof of Theorem 1 (in particular, from (2.6)) that $\rho(\theta, t)$ can be bounded from below by a universal positive constant independent of t_0 if the curve has initial non-negative flow constant $R(\theta, 0) \geqslant 0$. One could further prove that $\|R - \bar{R}\|_{L^\infty} \to 0$, which could yield that as $t \to \infty$, $\rho(\theta, t) \to \rho_\infty(\theta)$ with constant $\bar{R}$. It can be directly checked that the only 2π-periodic solution to $\bar{R} = constant$ for $\alpha > 4$ is $\rho_\infty(\theta) = constant$. Thus the limiting shape is a circle. On the other hand, for $\alpha = 4$, we know from Lemma 2 that $\rho_\infty(\theta)$ may not be constant.

3. Curve shortening type flows-revisit

In order to introduce non-adaptive flows for a convex, simple closed curve **X** in $\mathbb{R}^2$ with curvature function $k(\theta)$ ($\theta \in [0, 2\pi]$), for the time being we introduce a non-adaptive curvature $R_{\tau,\alpha}$ (similar to what we did in [9]) by

$$R_{\tau,\alpha} = k(\alpha \tau_{\theta\theta} + \tau),$$

where $\alpha > 0$ and τ is a function of k. We then introduce the following non-adaptive curve flow

$$k_t = R_{\tau,\alpha} k = k^2 (\alpha \tau_{\theta\theta} + \tau). \tag{3.1}$$

However, to assure that $k(\theta, t)$ will be a curvature of a simple closed curve along the flow, we need to choose $\alpha = 1$, since only in the case of $\alpha = 1$, we have

$$\partial_t \int_0^{2\pi} \frac{\cos\theta}{k} \, d\theta = -\int_0^{2\pi} \frac{\cos\theta}{k^2} \cdot k_t \, d\theta$$

$$= -\int_0^{2\pi} \cos\theta \cdot (\tau_{\theta\theta} + \tau) \, d\theta$$

$$= -\int_0^{2\pi} \tau \cdot \left((\cos\theta)_{\theta\theta} + \cos\theta\right) = 0$$

and

$$\partial_t \int_0^{2\pi} \frac{\sin\theta}{k} \, d\theta = 0,$$

which guarantees that, for all $t > 0$, the orthogonal condition is preserved:

$$\int_0^{2\pi} \frac{\sin\theta}{k} \, d\theta = \int_0^{2\pi} \frac{\cos\theta}{k} \, d\theta = 0.$$

We need to point out here that a π-periodic function always satisfies (1.7).

On the other hand, it can be shown (see, e.g., [7]) that flow

$$k_t = k^2(\tau_{\theta\theta} + \tau) \tag{3.2}$$

with the initial curvature satisfying (1.7) is equivalent to curve shortening flow:

$$\mathbf{X}_t = \tau \mathbf{N}$$

where $\mathbf{N}$ is the inner unit norm of the curve $\mathbf{X}$.

3.1. Homogeneous curve shortening flows

From the viewpoint of curvature flow, we shall give a direct proof of the global existence for homogeneous curve shortening flow (i.e. $\tau = k^p$) for $p > 1$. Throughout this subsection, we always assume that $p > 1$, and use

$$R_p = k\left((k^p)_{\theta\theta} + k^p\right) \tag{3.3}$$

to represent the $p - 1$-curve shortening flow curvature. The $p - 1$-curvature flow is defined as:

$$k_t = R_p k.$$

Introducing the average of curvature by

$$\overline{R}_p = \frac{\int_0^{2\pi} R_p \cdot k^{p-1}\, d\theta}{\int_0^{2\pi} k^{p-1}\, d\theta}, \tag{3.4}$$

we will consider its normalized curve shortening flow:

$$k_t = (R_p - \overline{R}_p)k. \tag{3.5}$$

Proposition 2. *For $p > 1$, if $k(\theta, t)$ satisfies (3.5) with a positive $k_0(\theta) \in C^1(S^1)$, then $k(\theta, t) > 0$ exists for all $t > 0$.*

Proof. Again, we only need to derive the positive lower bound and upper bound for $k(\theta, t)$. From the Strong Maximum Principle we know that $k(\theta, t) > 0$.

The main step in obtaining the upper bound for $k(\theta, t)$ is to show that for any given time $t_0 > 0$, $\int_0^{2\pi} k^{p-1}\, d\theta$, as well as $\int_0^{2\pi} [(k^p)_\theta]^2 - [k^p]^2\, d\theta$ are bounded for $t \in (0, t_0)$.

We first observe that flow (3.5) preserves $\int_0^{2\pi} k^{p-1}\, d\theta$:

$$\partial_t \int_0^{2\pi} k^{p-1}\, d\theta = (p-1) \int_0^{2\pi} (R_p - \overline{R}_p) k^{p-1}\, d\theta = 0.$$

Without loss of generality, we can assume that $\int_0^{2\pi} k^{p-1}\, d\theta = 2\pi$ for all $t > 0$. It follows that for any $\delta > 0$, there is a constant $C_\delta > 0$ such that $k \leqslant C_\delta$ except on intervals of length less than or equal to δ. We use the assumption of $p > 1$ here.

Also, we have, along the flow, that

$$\partial_t \int_0^{2\pi} R_p \cdot k^{p-1}\, d\theta = \partial_t \int_0^{2\pi} k^p\big((k^p)_{\theta\theta} + k^p\big)\, d\theta$$

$$= \int_0^{2\pi} (k^p)_t \big((k^p)_{\theta\theta} + k^p\big)\, d\theta + \int_0^{2\pi} k^p \big(((k^p)_t)_{\theta\theta} + (k^p)_t\big)\, d\theta$$

$$= 2 \int_0^{2\pi} (k^p)_t \big((k^p)_{\theta\theta} + k^p\big)\, d\theta$$

$$= 2p \int_0^{2\pi} k^{p-1} k_t \big((k^p)_{\theta\theta} + k^p\big)\, d\theta$$

$$= 2p \int_0^{2\pi} (R_p - \overline{R}_p) R_p k^{p-1}\, d\theta$$

$$= 2p \int_0^{2\pi} (R_p - \overline{R}_p)^2 k^{p-1}\, d\theta$$

$$\geqslant 0.$$

Note:

$$\int\limits_0^{2\pi} R_p \cdot k^{p-1} \, d\theta = \int\limits_0^{2\pi} \left(-\left[(k^p)_\theta \right]^2 + k^{2p} \right) d\theta.$$

So we know that there is a constant D depending only on k_0 such that

$$\int\limits_0^{2\pi} \left(\frac{\partial (k^p)}{\partial \theta} \right)^2 d\theta \leqslant \int\limits_0^{2\pi} k^{2p} \, d\theta + D. \tag{3.6}$$

Finally, we follow the proof of the pointwise estimate in Gage and Hamilton [7] to obtain the upper bound for curvature. Let $k(\psi_0) = \max_{[0,2\pi]} k(\theta)$, and $\psi \in [a, b]$ so that $k > C_\delta$ in interval (a, b) and $k(a) = C_\delta$. Then the length of $[a, b]$ is less than or equal to δ. For any $\psi \in [a, b]$,

$$
\begin{aligned}
k(\psi)^p &= k(a)^p + \int\limits_a^\psi \frac{\partial (k^p)}{\partial \theta} \, d\theta \\
&\leqslant C_\delta^p + \sqrt{\delta} \left(\int\limits_a^\psi \left(\frac{\partial (k^p)}{\partial \theta} \right)^2 d\theta \right)^{1/2} \\
&\leqslant C_\delta^p + \sqrt{\delta} \left(\int\limits_0^{2\pi} \left(\frac{\partial (k^p)}{\partial \theta} \right)^2 d\theta \right)^{1/2} \\
&\leqslant C_\delta^p + \sqrt{\delta} \left(\int\limits_0^{2\pi} k^{2p} \, d\theta + D \right)^{1/2}.
\end{aligned}
$$

It follows that for $k_{MAX} = \max_{t \in [0,2\pi]} k(\theta, t)$,

$$k_{MAX}^p \leqslant C_\delta^p + \sqrt{2\pi} \sqrt{\delta} k_{MAX}^p + \sqrt{2\pi \delta D}.$$

Choosing δ small enough we derive that

$$k_{MAX} < 2C_\delta + (2\delta D)^{\frac{1}{2p}}. \tag{3.7}$$

To show that k is bounded from below, we observe that

$$\overline{R}_p = \int\limits_0^{2\pi} k^{2p} - (k^p)_\theta^2 \, d\theta \leqslant D_1. \tag{3.8}$$

Let

$$\tilde{k} = k \cdot e^{\int_0^t \overline{R}_p \, ds}.$$

$\tilde{k}$ satisfies

$$\tilde{k}_t = e^{-(p+1)\int_0^t \bar{R}_p\, ds} \cdot \big(\tilde{k}^2 \big(\tilde{k}^p\big)_{\theta\theta} + (\tilde{k})^{p+2}\big).$$

The Maximum Principle yields that $\min_{\theta \in [0,2\pi]} \tilde{k}(\theta, t)$ is monotonically non-decreasing in t. Combining this with (3.7), we have: for any $t_0 > 0$,

$$k(\theta, t_0) \geqslant \min k_0(\theta) \cdot e^{-\int_0^{t_0} \bar{R}_p\, ds} \geqslant C(t_0). \tag{3.9}$$

From (3.7) and (3.9) we know the solution exists for all time t to equation

$$k_t = k^2 \big(k^p\big)_{\theta\theta} + k^{p+2} - \bar{R}_p k,$$

which is the same as (3.5). $\square$

We shall discuss the limiting shape under the flow at the end of this section.

Remark 3. The proof does not work for $p = 1$. Even though we can show that $\int_0^{2\pi} \ln k\, d\theta = constant$ under the normalized flow for $p = 1$, we cannot show that $k(\theta)$ is bounded from below from the equation. Without the assumption that $k(\theta, t)$ satisfies (1.7), we doubt that $k(\theta, t)$ is bounded for fixed $t \in (0, \infty)$. On the other hand, we know that singularity does arise for an immersion convex curve under curve shortening flow for $p = 1$, see for example, Angenent [1]. It is interesting to know whether one can show, for $p = 1$ and $k(\theta, t)$ satisfying (1.7), that $k(\theta, t)$ is bounded via a similar argument to the proof of Proposition 5 below, thus to recover the standard global existence result of Gage and Hamilton [7].

3.2. Non-homogeneous curve shortening flows

More generally, one considers the following non-homogeneous curve shortening flow. Define the curvature

$$R_{p_\lambda} = k\big((k^p)_{\theta\theta} + k^p\big) + \lambda k = k\big((k^p + \lambda)_{\theta\theta} + (k^p + \lambda)\big),$$

where λ is a parameter. The corresponding curvature flow is given by

$$k_t = R_{p_\lambda} k = k^2 \big((k^p)_{\theta\theta} + k^p\big) + \lambda k^2.$$

Define

$$\bar{R}_{p_\lambda} = \frac{\int_0^{2\pi} R_{p_\lambda} k^{p-1}\, d\theta}{\int_0^{2\pi} k^{p-1}\, d\theta}.$$

For $p > 1$ and $\lambda \geqslant 0$, we will show the global existence to the above flow (Theorem 2) via proving the global existence to the normalized flow

$$k_t = (R_{p_\lambda} - \bar{R}_{p_\lambda}) k. \tag{3.10}$$

Proposition 3. *For $p > 1$ and $\lambda \geqslant 0$, if $k(\theta, t)$ satisfies (3.10) with a positive $k_0(\theta) = k(\theta, 0) \in C^1(S^1)$, then $k(\theta, t) > 0$ exists for all $t > 0$.*

The proof is quite similar to that for homogeneous curve shortening flow. We shall just sketch it here.

Proof of Proposition 3. Since $\lambda \geqslant 0$, from the Strong Maximum Principle we know that $k(\theta, t) > 0$. Again we first observe that flow (3.10) preserves $\int_0^{2\pi} k^{p-1} \, d\theta$:

$$\partial_t \int\limits_0^{2\pi} k^{p-1} \, d\theta = (p-1) \int\limits_0^{2\pi} (R_{p_\lambda} - \overline{R}_{p_\lambda}) k^{p-1} \, d\theta = 0.$$

Without loss of generality, we assume that $\int_0^{2\pi} k^{p-1} \, d\theta = 2\pi$ for all $t > 0$. It follows that for any $\delta > 0$, there is a constant $C_\delta > 0$ such that $k \leqslant C_\delta$ except on intervals of length less than or equal to δ. The assumption of $p > 1$ is used here. Consider the energy

$$F_\lambda(k) := \int\limits_0^{2\pi} k^p \big((k^p)_{\theta\theta} + k^p - 2\lambda \big) \, d\theta = \int\limits_0^{2\pi} \big\{ k^{2p} - 2\lambda k^p - \big[(k^p)_\theta \big]^2 \big\} \, d\theta.$$

Along the flow we have

$$\partial_t F(k) = \partial_t \int\limits_0^{2\pi} \big\{ k^{2p} - 2\lambda k^p - \big[(k^p)_\theta \big]^2 \big\} \, d\theta$$

$$= \int\limits_0^{2\pi} \big\{ 2p k_t k^{2p-1} - 2\lambda p k_t k^{p-1} - 2(k^p)_\theta (k^p)_{t\theta} \big\} \, d\theta$$

$$= 2p \int\limits_0^{2\pi} k_t k^{p-1} \big\{ k^p - \lambda + (k^p)_{\theta\theta} \big\} \, d\theta$$

$$= 2p \int\limits_0^{2\pi} (R_{p_\lambda} - \overline{R}_{p_\lambda}) R_{p_\lambda} k^{p-1} \, d\theta$$

$$= 2p \int\limits_0^{2\pi} (R_{p_\lambda} - \overline{R}_{p_\lambda})^2 k^{p-1} \, d\theta$$

$$\geqslant 0.$$

Thus, there is a constant D_2 depending only on k_0 such that

$$\int\limits_0^{2\pi} \left(\frac{\partial (k^p)}{\partial \theta} \right)^2 \, d\theta \leqslant \int\limits_0^{2\pi} k^{2p} \, d\theta + D_2. \tag{3.11}$$

From here, one can show that k is bounded from above and from below by a positive constant, similar to the proof of Proposition 2. This yields the global existence for the flow. $\quad\square$

3.3. Limiting shapes

If k is the curvature of a simple and closed convex curve, (3.10) is equivalent to non-homogeneous curve shortening flow

$$\mathbf{X}_t = (k^p + \lambda)\mathbf{N}. \tag{3.12}$$

Let $\tau = k^p + \lambda$. Consider its geometric normalized flow

$$\mathbf{X}_t = (\tau - \tau_a)\mathbf{N}, \tag{3.13}$$

where $\tau_a = \int_0^L \tau \, ds / L = \int_0^{2\pi} k^p \, ds / L + \lambda$. Then it is an area-preserving flow:

$$\partial_t (area) = -\int_0^L (\tau - \tau_a) \, ds = 0.$$

On the other hand, we observe that

$$\partial_t (length) = -\int_0^L (\tau - \tau_a) k \, ds$$

$$= -\int_0^L k^{p+1} \, ds + \frac{\int_0^L k^p \, ds \cdot \int_0^L k \, ds}{L}$$

$$\leqslant 0.$$

Thus, the isoperimetric constant is decreasing along the flow, which indicates the limit shape to the geometric normalized flow (3.13) will be a circle. However, this does not indicate that the limiting shape for flow (3.10) is also a circle. In fact, when $p = 1/3$, $\lambda = 0$, the above argument indicates that limiting shape for (3.13) is a circle, but the limiting shape for (3.10) (which is the normalized affine curvature flow) in fact is known to be an elliptic point.

4. New adaptive flows

Our understanding of curve flow problem from curvature flow equation prompts us to consider a new adaptive flows as follows.

For any positive, 2π-periodic function $\rho(\theta) \in C^2[0, 2\pi]$, and a given positive parameter α, we define its (α, τ)-*shorten-flow constant* by

$$R_\tau^\alpha(\rho) = \rho(\alpha(\tau)_{\theta\theta} + \tau),$$

where τ is a function of ρ. The average (α, τ)-*shorten-flow constant* is given by

$$\overline{R}_\tau^\alpha = \frac{\int_0^{2\pi} R_\tau^\alpha \cdot \rho^{p-1} \, d\theta}{\int_0^{2\pi} \rho^{p-1} \, d\theta}.$$

Then the new adaptive motion equation can be defined as

$$\rho_t = \left(R_\tau^\alpha - \overline{R}_\tau^\alpha\right)\rho.$$

Here we shall just focus on a concrete example: $\tau = \rho^p$ for $p > 0$, and call the corresponding (α, τ)-*shorten-flow constant* the (α, p)-*flow constant*:

$$R_p^\alpha(\rho) = \rho\left(\alpha\left(\rho^p\right)_{\theta\theta} + \rho^p\right). \qquad (4.1)$$

The average (α, p)-*flow constant* is given by

$$\overline{R}_p^\alpha = \frac{\int_0^{2\pi} R_p^\alpha \cdot \rho^{p-1}\,d\theta}{\int_0^{2\pi} \rho^{p-1}\,d\theta}, \qquad (4.2)$$

and the adaptive motion equation is defined as

$$\rho_t = \left(R_\rho^\alpha - \overline{R}_\rho^\alpha\right)\rho, \quad \text{that is} \quad \rho_t = \alpha\rho^2\left(\rho^p\right)_{\theta\theta} + \rho^{p+2} - \overline{R}_p^\alpha \rho. \qquad (4.3)$$

Our current methods enable us to obtain the global existence to the above equation for $p \geqslant \frac{1}{3}$ for certain range of α.

4.1. The case of $p > 1$

We first prove:

Theorem 3. *For $\alpha > 0$ and $p > 1$, if $\rho(\theta, t)$ satisfies (4.3) with $\rho(\theta, 0) = \rho_0(\theta)$, where $\rho_0(\theta) \in C^1[0, 2\pi]$ is a positive, 2π-periodic function, then $\rho(\theta, t)$ exists for all $t > 0$.*

Proof. The proof is almost the same as that of Proposition 2. The Strong Maximum Principle yields that $\rho(\theta, t) > 0$.

First, observe that flow (4.3) preserves $\int_0^{2\pi} \rho^{p-1}\,d\theta$:

$$\partial_t \int_0^{2\pi} \rho^{p-1}\,d\theta = (p-1)\int_0^{2\pi} \left(R_p^\alpha - \overline{R}_p^\alpha\right)\rho^{p-1}\,d\theta = 0.$$

Without loss of generality, we can assume that $\int_0^{2\pi} \rho^{p-1}\,d\theta = 2\pi$ for all $t > 0$. Next, we only need to show that $\int_0^{2\pi} \alpha[(\rho^p)_\theta]^2 - [\rho^p]^2\,d\theta$ are bounded for $t \in (0, t_0)$. Then, similar to the proof of Proposition 2, from these we can show that ρ is bounded from below and above by some positive constants for fixed time $t_0 > 0$, which yields the global existence of solution.

Along the flow, we have

$$\partial_t \int_0^{2\pi} R_p^\alpha \cdot \rho^{p-1}\,d\theta = \partial_t \int_0^{2\pi} \rho^p\left(\alpha\left(\rho^p\right)_{\theta\theta} + \rho^p\right)d\theta$$

$$= \int_0^{2\pi} \left(\rho^p\right)_t\left(\alpha\left(\rho^p\right)_{\theta\theta} + \rho^p\right)d\theta + \int_0^{2\pi} \rho^p\left(\left(\alpha\left(\rho^p\right)_t\right)_{\theta\theta} + \left(\rho^p\right)_t\right)d\theta$$

$$= 2 \int_0^{2\pi} (\rho^p)_t (\alpha(\rho^p)_{\theta\theta} + \rho^p) \, d\theta$$

$$= 2p \int_0^{2\pi} \rho^{p-1} \rho_t (\alpha(\rho^p)_{\theta\theta} + \rho^p) \, d\theta$$

$$= 2p \int_0^{2\pi} (R_p - \overline{R}_p^\alpha) R_p \rho^{p-1} \, d\theta$$

$$= 2p \int_0^{2\pi} (R_p - \overline{R}_p^\alpha)^2 \rho^{p-1} \, d\theta$$

$$\geqslant 0.$$

Note:

$$\int_0^{2\pi} R_p \cdot \rho^{p-1} \, d\theta = \int_0^{2\pi} \left(-\alpha \big[(\rho^p)_\theta \big]^2 + \rho^{2p} \right) d\theta.$$

So we know that there is a constant D_3 depending only on k_0 such that

$$\int_0^{2\pi} \left(\alpha \big[(\rho^p)_\theta \big]^2 - \rho^{2p} \right) d\theta \leqslant D_3. \qquad \square$$

4.2. Limiting shapes

Since ρ is an arbitrary positive, 2π-periodic function, we cannot expect to show that the flow will converge to circle via isoperimetric inequality. Rather, the limiting shapes, if the flow converges, are complicated. The main difficulty is due to the lack of understanding of 2π-periodic positive solutions to the following equation

$$\alpha u_{\theta\theta} + u = u^{-\frac{1}{p}} \quad \text{on } S^1. \tag{4.4}$$

Let $\phi(\theta) = u(\sqrt{\alpha}\theta)$. Eq. (4.4) can be reduced to

$$\phi_{\theta\theta} + \phi = \phi^{-\frac{1}{p}}, \qquad \phi(\theta + 2\pi/\sqrt{\alpha}) = \phi(\theta). \tag{4.5}$$

Eq. (4.5) was discussed by Andrews [3] when $\sqrt{\alpha}$ is a rational number. In particular, he showed that there are non-constant solutions to Eq. (4.5). For $p > 1$ and any α close to zero, we will show that most likely there are *non-constant solutions*.

For $u \in H^1(S^1)$, we define

$$\mathcal{F}_{\alpha,p}(u) = \int_{S^1} \left(u^{2p} - \alpha (u^p)_\theta^2 \right) d\theta$$

and

$$\Gamma_{p,2\pi} = \left\{ u \in H^1(S^1) \colon \int_0^{2\pi} u^{p-1}\, d\theta = 2\pi \right\}.$$

From the proof of Theorem 4 we know that along flow (4.3), if initial function $\rho_0 \in \Gamma_{p,2\pi}$, then $\rho \in \Gamma_{p,2\pi}$, and energy $\mathcal{F}_{\alpha,p}(\rho)$ is monotonically increasing. Moreover, we know that ρ is uniformly bounded with respect to time t, thus $\mathcal{F}_{\alpha,p}(\rho)$ is bounded above. These shall imply (we hope to address the convergence more rigorously later in a future paper) that $\rho(\theta, t) \to \rho_*(\theta)$ in a suitable sense, where

$$R_p^\alpha(\rho_*) = \overline{R}_p^\alpha = constant.$$

On the other hand, if $\rho_0 \in \Gamma_{p,2\pi}$ is not a constant, then we know that

$$\int_0^{2\pi} \rho_0^{2p}\, d\theta > \left(\int_0^{2\pi} \rho_0^{p-1}\, d\theta \right)^{\frac{2p}{p-1}} \cdot 2\pi^{\frac{p+1}{p-1}} = 2\pi.$$

Thus, there is an $\alpha_0 > 0$, such that for $\alpha \in (0, \alpha_0)$, $\mathcal{F}_{\alpha,p}(\rho_0) > 2\pi$, thus $\mathcal{F}_{\alpha,p}(\rho_*) > 2\pi$. This indicates that the limit ρ_* cannot be constant since otherwise $\rho_* = 1$ from Eq. (4.4), which yields $\mathcal{F}_{\alpha,p}(\rho_*) = 2\pi$.

For general $\alpha \in (0, 4)$, one may find ρ_0 given by

$$\rho_0^p = c_p \sqrt{\lambda^2 \cos^2 \frac{\theta}{2} + \lambda^{-2} \sin^2 \frac{\theta}{2}},$$

for suitable $\lambda > 0$ and c_p so that $\rho_0 \in \Gamma_{p,2\pi}$ and $\mathcal{F}_{\alpha,p}(\rho) > 2\pi$. With such initial data, the limiting of ρ could not be constant if it converges. We shall not pursue the details here.

4.3. The case of $p \in [1/3, 1)$

We shall show that the proof of Proposition 1 can be adapted to establish

Proposition 4. *For $\alpha \geqslant 4$ and $p \in [1/3, 1)$, if $\rho(\theta, t)$ satisfies (4.3) with $\rho(\theta, 0) = \rho_0(\theta)$, where $\rho_0(\theta) \in C^1[0, 2\pi]$ is a positive, 2π-periodic function, then $\rho(\theta, t)$ exists for all $t > 0$.*

Proof. Again, we only need to show that for any given $t_0 > 0$, there is a positive constant $C_4 = C_4(t_0)$ depending on t_0, such that

$$\frac{1}{C_4(t_0)} \leqslant \rho(\theta, t) \leqslant C_4(t_0).$$

Since the flow preserves $\int_0^{2\pi} \rho^{p-1}\, d\theta$, without loss of generality, we assume that $\int_0^{2\pi} \rho^{p-1}\, d\theta = 1$. We first derive the lower bound: Let

$$\tilde{\rho} = \rho \cdot e^{\int_0^t \overline{R}_p^\alpha\, ds}.$$

Then $\tilde{\rho}$ satisfies

$$\tilde{\rho}_t = e^{-(p+1)\int_0^t \bar{R}_p^\alpha \, ds} \cdot \left(\tilde{\rho}^2 (\tilde{\rho}^p)_{\theta\theta} + (\tilde{\rho})^{p+2}\right).$$

The Maximum Principle yields that $\min_{\theta \in [0,2\pi]} \tilde{k}(\theta, t)$ is monotonically non-decreasing in t, thus for any $t_0 > 0$,

$$\rho(\theta, t) \geqslant \min \rho_0(\theta) \cdot e^{-\int_0^{t_0} \bar{R}_p^\alpha \, ds}, \quad \forall t \in [0, t_0].$$

On the other hand, since $\alpha \geqslant 4$, using Corollary 1 and the Hölder inequality, we have

$$\bar{R}_p^\alpha = \int_0^{2\pi} (\rho^p)^2 - \alpha \left[(\rho^p)_\theta\right]^2 d\theta$$

$$\leqslant 4\pi^2 \left(\int_0^{2\pi} (\rho^p)^{-2} \, d\theta\right)^{-1} \quad \text{(here we use Corollary 1)}$$

$$\leqslant C \left(\int_0^{2\pi} \rho^{p-1} \, d\theta\right)^{\frac{2p}{p-1}} \quad \text{(here we use the fact that } p \geqslant 1/3)$$

$$= C.$$

So we obtain the lower bound for $\rho(\theta, t)$.

To obtain the upper bound, we first observe that for all $\alpha > 0$, along the flow,

$$\partial_t \int_0^{2\pi} R_p^\alpha \cdot \rho^{p-1} \, d\theta = 2p \int_0^{2\pi} \left(R_p - \bar{R}_p^\alpha\right)^2 \rho^{p-1} \, d\theta \geqslant 0.$$

Thus,

$$\int_0^{2\pi} (\rho^p)^2 - \alpha \left[(\rho^p)_\theta\right]^2 d\theta \geqslant \bar{R}_p^\alpha(0), \tag{4.6}$$

where $\bar{R}_p^\alpha(0) = \bar{R}_p^\alpha(t)|_{t=0}$.

Similar to the proof of Proposition 1, since ρ is bounded from below by a positive constant and $\int_0^{2\pi} \rho^{p-1} \, d\theta = 1$, we know that there is a constant $\delta_1(t_0) > 0$, such that

$$\left|\left\{\theta : \rho(\theta, t) \leqslant 2\right\}\right| > \delta_1(t_0), \quad \text{for } t \in [0, t_0]. \tag{4.7}$$

Similar to the proof of Proposition 1, from (4.6) and (4.7), we can show that $\int_0^{2\pi} \rho^{2p} \, d\theta$ is bounded. In turn, from (4.6) we know that $\int_0^{2\pi} [(\rho^p)_\theta]^2 \, d\theta$ is bounded. Thus ρ is bounded due to Sobolev embedding $H^1(S^1) \hookrightarrow C^{0,\frac{1}{2}}(S^1)$. $\quad\square$

4.4. The case of $p = 1$

For $p = 1$, we define the average total curvature as

$$\overline{R}_1^\alpha = \int\limits_0^{2\pi} R_1^\alpha \, d\theta,$$

and consider the normalized flow

$$\rho_t = \left(R_1^\alpha - \overline{R}_1^\alpha\right)\rho. \tag{4.8}$$

We will show

Proposition 5. *For $\alpha \geqslant 4$ if $\rho(\theta, t)$ satisfies (4.8) with $\rho(\theta, 0) = \rho_0(\theta)$, where $\rho_0(\theta) \in C^1[0, 2\pi]$ is a positive, 2π-periodic function, then $\rho(\theta, t)$ exists for all $t > 0$.*

Proof. First, we check that

$$\partial_t \int\limits_0^{2\pi} \ln \rho \, d\theta = \int\limits_0^{2\pi} \left(R_1^\alpha - \overline{R}_1^\alpha\right) d\theta = 0.$$

Without of loss of generality, we can assume that $\int_0^{2\pi} \ln \rho \, d\theta = 1$.

Using Corollary 1, we have

$$\overline{R}_1^\alpha = \int\limits_0^{2\pi} \rho^2 - \alpha \rho_\theta^2 \, d\theta \leqslant 4\pi^2 \left(\int\limits_0^{2\pi} \rho^{-2} \, d\theta \right)^{-1}$$

$$\leqslant C_1 \cdot \exp\left\{ C_2 \int\limits_0^{2\pi} \ln \rho \, d\theta \right\}$$

$$\leqslant C_3.$$

Similar to the proof of Theorem 4, the upper bound for $\overline{R}_1^\alpha$ yields the positive lower bound for ρ.

Also, along the flow,

$$\partial_t \int\limits_0^{2\pi} R_1^\alpha \, d\theta = \partial_t \int\limits_0^{2\pi} \rho(\alpha \rho_{\theta\theta} + \rho) \, d\theta$$

$$= \int\limits_0^{2\pi} \rho_t(\alpha \rho_{\theta\theta} + \rho) \, d\theta + \int\limits_0^{2\pi} \rho(\alpha \rho_{t\theta\theta} + \rho_t) \, d\theta$$

$$= 2 \int\limits_0^{2\pi} \rho_t(\alpha \rho_{\theta\theta} + \rho) \, d\theta$$

$$= 2 \int_0^{2\pi} \left(R_1 - \overline{R}_1^{\alpha} \right) R_1 \, d\theta$$

$$= 2 \int_0^{2\pi} \left(R_p - \overline{R}_p^{\alpha} \right)^2 d\theta$$

$$\geqslant 0.$$

Thus

$$\int_0^{2\pi} \rho^2 - \alpha \rho_\theta^2 \, d\theta \geqslant \overline{R}_p^{\alpha}(0), \tag{4.9}$$

where $\overline{R}_1^{\alpha}(0) = \overline{R}_1^{\alpha}(t)|_{t=0}$. Since ρ is bounded from below by a positive constant and $\int_0^{2\pi} \ln \rho \, d\theta = 1$, we have

$$\left| \left\{ \theta \colon \rho(\theta, t) \leqslant 2 \right\} \right| > \delta_2(t_0), \quad \text{for } t \in [0, t_0] \tag{4.10}$$

for some positive constant $\delta_2(t_0)$. We then can derive the upper bound for ρ from (4.9) and (4.10). The global existence then follows from the standard parabolic estimates. $\quad\square$

The proof of Theorem 4 follows from the combining of Propositions 4 and 5.

Acknowledgments

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