Review for the final exam

Final Exam: May 15, 1:30-3:30pm at PHSC 222

Office hour: Monday and Tuesday (May 11, May 12) 10:00–12:30pm
Chapter 1: 1.1-1.6; Chapter 3: 3.1-3.3, 3.5, 3.8; and Chapter 4: 4.1; Total count 30%

Chapter 7: 7.1-7.4. Chapter 8: 8.1-8.2. 70%.

1 First order ODEs

Definition: We start with some terminologies: Differential equations, solutions, general solutions, particular solutions, etc

Exercise 1.1: Check $y = e^{-2x}$ solves

$$y'' - 4y = 0.$$

How to solve a differential equation (1st order): 7 types:

Type 1:

$$\frac{dy}{dx} = f(x).$$

We learned it in Calculus course.

Type 2:

$$\frac{dy}{dx} = g(x)h(y).$$

Separable equations! Change it to

$$\frac{dy}{h(y)} = g(x)dx.$$

Caution: $h(y) = 0$ **might be an** *orphan* **solution!**

Exercise 1.2: Solve

$$\frac{dy}{dx} + x = xy^2.$$

Type 3:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Linear. Find integrating factor.

Exercise 1.3: Solve

$$\frac{dy}{dx} + xy = x.$$

Type 4:

$$\frac{dy}{dx} = F(ax + by + c).$$

Linear substitution: let $v(x) = ax + by + c$.

Exercise 1.4: Solve

$$\frac{dy}{dx} = (x + y + 2)^2.$$

Type 5:

$$\frac{dy}{dx} = F(\frac{y}{x}).$$

Nonlinear substitution: let $v(x) = y/x$.

Exercise 1.5: Solve

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2.$$

Type 6:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Bernoulli equations. Change it to

$$\frac{dy}{y^n dx} + P(x)y^{-(n-1)} = Q(x).$$

Then let $v(x) = y^{-(n-1)}$.

Exercise 1.6: Solve

$$y^2\frac{dy}{dx} + 2xy^3 = 6x.$$

Type 7:

$$M(x,y)dx + N(x,y)dy = 0.$$

Exact equations.

Exercise 1.7: Solve

$$(x^3 + \frac{y}{x})dx + (y^2 + lnx)dy = 0.$$

2 Higher order ODEs

Definition: We start with some terminologies: Linear differential equations, superposition, general solutions, homogeneous and nonhomogeneous equations, particular solutions, etc

Exercise 2.1: If y_1 and y_2 solve equation

$$y^{''} + e^x y' + y \sin x^2 = 0,$$

check $2y_1 + 3y_2$ is also a solution.

How to solve a homogeneous equation of **constant coefficients**:

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = 0.$$

Three types.

Consider the Characteristic equation.

Type 1: Distinct roots $s_1, ..., s_n$

Type 2: Repeat roots (roots with multiplicity greater than 1)

Exercise 2.2: Find and check the general solution to

$$y'' + 4y' + 4 = 0.$$

Type 3: complex roots.

Exercise 2.3: If the characteristic equation of a ODE is

(a).

$$(r-1)^3(r^2 - 4r + 5) = 0,$$

(b).

$$(r-1)^3(r^2 - 2r + 2)^2 = 0,$$

find the general solutions.

How to solve a nonhomogeneous equation of constant coefficients:

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = f(x).$$

Undetermined coefficient method to find one particular solution.

Exercise 2.4: Find the general solution to

$$y'' + 4y' + 5y = x + e^{2x} \sin x.$$

Variation of parameter method: the formula:

$$y(x) = -y_1 \int \frac{y_2 f(x)}{W(y_1, y_2} dx + y_2 \int \frac{y_1 f(x)}{W(y_1, y_2} dx.$$

Eigenvalue problems

Exercise 2.5: Find the first positive eigenvalue to

$$-y'' = \lambda y, \quad y'(0) = y'(\pi) = 0.$$

3 Laplace transform

Laplace transform: Definition and basic formulas for e^{at}, $\cos kt$, $\sin kt$, t^a, $\cosh kt$, $\sinh kt$, $f'(t)$, translation and partial fraction.

Exercise 3.1: Find: $\mathcal{L}^{-1}\{F(s)\}$ if
(a). $F(s) = \frac{1}{s(s-3)}$;
(b). $F(s) = \frac{1}{s^2(s-3)}$;
(c). $F(s) = \frac{s-1}{(s+1)^3}$ (**hint: translation!**).

Using Laplace transform to solve initial value problem: Differential equation to Algebraic equation, solving Algebraic equation, then find $\mathcal{L}^{-1}\{F(s)\}$.

Exercise 3.2: Solve:
(a). $x'' + 4x = \sin 2t, \quad x(0) = x'(0) = 0.$
(b). $x' = 4x + 2y, y' = 3x - y; x(0) = 3, \ y(0) = -2.$

Advanced tricks for laplace transform: Convolution, derivative and integral of transforms.

Exercise 3.3: Find $\mathcal{L}^{-1}\{F(s)\}$ if
(a). $F(s) = \frac{1}{(s^2+9)^2}$;
(b). $F(s) = \ln \frac{s-2}{s+2}$.

Exercise 3.4: Find a nontrivial solution such that $x(0) = 0$ to

$$tx'' + (t-2)x' + x = 0.$$

Can you find another solution? (**Hard, not for everyone**).

4 Power series method

Find the recurrence relation, and estimate the radius of convergence. Need to remember Taylor series for e^x, $\cos x$, $\sin x$, $\frac{1}{1-x}$, $\ln(1+x)$.

Exercise 4.1 Show that power series method fails to solve

$$x^2 y' = y - x - 1.$$

Exercise 4.2: Find one polynomial solution to

$$(1-x^2)y'' - 2xy' + 6y = 0.$$

For a power series solution

$$y = \sum_{n=0}^{\infty} c_n x^n$$

will it converge for $x = 1/2$?