

Review for Midterm

Extra office hour: Wed 2:30-3:30pm

On Preliminaries (Chapter 1).

Set and functions: De Morgan Law, operations for sets, functions.

Exercise 1: Find

$$\bigcap_{i=1}^{\infty} A_i$$

when

(a). $A_i = [0, \frac{1}{n}]$, (b). $A_i = [0, \frac{1}{n})$, (c). $A_i = (0, \frac{1}{n}]$, (d). $A_i = (0, \frac{1}{n})$

Math induction: standard and strong induction, why it works.

Exercise 2: (a). Prove the "Principle of Strong induction" on page 15.

(b). Prove that $5^{2n} - 1$ is divisible for all natural number n .

Finite and infinite sets: Definitions. Existence of uncountable sets (relation to Nested Interval Property) and Cantor's theorem.

Exercise 3: Is there any bijection between the set of all positive rational number and the set of natural number? If yes, give an example.

On Real numbers (Chapter 2).

Basic algebraic properties: Algebraic operations and Absolute values.

Exercise 4. Let $f(x)$ be a function defined in an interval I. Show that there are an odd function $g(x)$ and an even function $h(x)$ in I such that

$$f(x) = g(x) + h(x).$$

Completeness and Supremum (infimum): Completeness axiom, Supremum and Infimum, Archimedean principle, density, intervals.

Exercise 5:

(a). Let A be a set of some positive numbers. Prove that

$$\inf \frac{1}{A} = \frac{1}{\sup A}.$$

where $\frac{1}{A} = \{\frac{1}{a} : a \in A\}$.

(b). If $\lim x_n = \sup\{x_n : n \in N\}$, is x_{n+m} an increasing sequence for large natural number m ?

On Sequences (Chapter 3). limit: Definition, operation, convergence, divergence.

Exercise 6: Using definition to prove

$$\lim \sqrt{1 + \frac{1}{n}} = 1.$$

Monotone and subsequences Convergence for monotonic and bounded sequences, subsequences

Exercise 7: (a). Let a be a positive number. If $x_1 \in (0, 1/a)$ and

$$x_{n+1} = x_n(2 - ax_n),$$

show that $\lim x_n$ exists. What is it?

(b). (Hardest one) If $x_1 = 1$ and

$$x_{n+1} = \frac{1}{1 + x_n},$$

prove $\lim x_n = (\sqrt{5} - 1)/2$. (That is: prove the existence of limit first, then compute the limit.)

Proof of part b):

Note

$$x_{2n+2} = \frac{1 + x_{2n}}{2 + x_{2n}},$$

and

$$x_{2n+1} = \frac{1 + x_{2n-1}}{2 + x_{2n-1}}.$$

One can show that

$$x_{2n+2} - x_{2n} = \frac{x_{2n} - x_{2n-2}}{(2 + x_{2n})(2 + x_{2n-2})},$$

and

$$x_{2n+1} - x_{2n-1} = \frac{x_{2n-1} - x_{2n-3}}{(2 + x_{2n-1})(2 + x_{2n-3})}.$$

From these, we can show x_{2n} is decreasing and bound above by x_0 , below by 0; And x_{2n+1} is increasing and bounded above by 1, below by 0. Thus, both of them converge, and the limit is the positive solution to

$$x = \frac{1}{1+x},$$

thus it is $\frac{\sqrt{5}-1}{2}$.

On Sequences and series (Chapter 3). Bolzano-Weierstrass Theorem, Cauchy criterion, convergence or divergence for sequences and series.

Exercise 8:

(a). If $\{x_n\}$ is a monotone increasing sequence that is bounded above by M . Prove that $\lim x_n$ exists.

(b). Give examples to show that a bounded sequence may not be a Cauchy sequence. Will a Cauchy sequence be bounded?

Exercise 9: Let

$$\sum_{n=1}^{\infty} a_n$$

be a convergent series and $a_n > 0$. If $b_k = \frac{a_1+a_2+\dots+a_k}{k}$, then

(a).

$$\lim b_k = 0.$$

(b). Will $\sum_{k=1}^{\infty} b_k$ be a convergent or a divergent series?

Limits for functions (Chapter 4). Definition, the link between limits of functions and limits of sequences.

Exercise 10: Show that $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ does not exist.

WARNING: YOU ARE RESPONSIBLE FOR CHECKING OUT MY TYPOS!

Comments and question to: mzhu@math.ou.edu

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