Solution

Exercise 3

Solution :

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n t + \sum_{n=1}^{\infty} b_n \sin n t .$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos n t \, dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} \frac{t}{2} \cos n t \, dt + \frac{1}{\pi} \int_{0}^{\pi} \frac{t}{2} \cos n t \, dt$$

$$= \frac{1}{\pi} \int_{\pi}^{0} \frac{t}{2} \cos n(-t) \, dt + \frac{1}{\pi} \int_{0}^{\pi} \frac{t}{2} \cos n t \, dt$$

$$= -\frac{1}{\pi} \int_{0}^{\pi} \frac{t}{2} \cos n t \, dt + \frac{1}{\pi} \int_{0}^{\pi} \frac{t}{2} \cos n t \, dt$$

$$= 0 .$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin n t \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{t}{2} \sin n t \, dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{t}{2} \frac{d \cos n t}{(-n)} = \frac{1}{\pi} \cdot \frac{t}{2} \cdot \frac{\cos n t}{-n} \Bigg|_{-\pi}^{\pi} - \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos n t}{2(-n)} dt$$

$$= -\frac{\cos n\pi}{n} = \frac{(-1)^{n+1}}{n} .$$

So $\frac{t}{2} \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt$.

Exercise 4.

Solution: 1) $\lambda < 0$, Let $\lambda = -k^2$ for $k > 0$.

General solution for $y'' - k^2 y = 0$ is

$$y = c_1 e^{kx} + c_2 e^{-kx}$$

$$y'(x) = c_1 k e^{kx} - c_2 k e^{-kx}$$

$y'(0) = 0 \Rightarrow 0 = c_1 k - c_2 k$, so $c_1 = c_2$

$y(L) = 0 \Rightarrow c_1 e^{kL} + c_2 e^{-kL} = 0$

$\Rightarrow c_1 (e^{kL} + e^{-kL}) = 0$, thus $c_1 = 0$

so $c_2 = 0$

No eigenvalue.

2) $\lambda = 0$, General solution is

$$y = c_1 + c_2 x$$

$y'(x) = c_2$, $y'(0) = 0 \Rightarrow c_2 = 0$

$y(L) = 0 \Rightarrow c_1 = 0$. no eigenvalue

3). $\lambda > 0$, Let $\lambda = k^2$ for $k > 0$

then General solution to $y'' + k^2 y = 0$ is

$$y(x) = c_1 \cos kx + c_2 \sin kx$$

$$y'(x) = -c_1 k \sin kx + c_2 k \cos kx$$

$y'(0) = 0 \implies c_2 k = 0$, thus $c_2 = 0$

$y(L) = 0 \implies c_1 \cos kL = 0$, so $kL = \frac{\pi}{2} + n\pi$

$$k = \frac{(2n+1)\pi}{2L}.$$

$\lambda = \left(\frac{(2n+1)\pi}{2L}\right)^2$ are eigenvalues for $n = 0, 1, \dots$

$y_n = \cos \frac{(2n+1)\pi x}{2L}$ are corresponding eigenfunctions.

Exercise 5.

Solution

step 1 Solve $\begin{cases} u_t = u_{xx} \\ u_x(0,t) = u(L,t) = 0 \end{cases}$ $(*)$

Let $u(x,t) = X(x)T(t)$, then

$$XT' = X''T \implies \frac{T'}{T} = \frac{X''}{X} = -\lambda$$

$$u_x(0,t) = u(L,t) = 0 \implies X'(0) = X(L) = 0.$$

Need to solve eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X(L) = 0. \end{cases}$$

By exercise 4,
$$\lambda_n = \left[\frac{(2n+1)\pi}{2L}\right]^2, \qquad n = 0, 1, \ldots$$

$$X_n(x) = \cos\frac{(2n+1)\pi x}{2L}.$$

For such λ_n, $T' = -\lambda_n T$

$$T_n(t) = e^{-\lambda_n t} = e^{-\left[\frac{(2n+1)\pi}{2L}\right]^2 t}, \qquad n = 0, 1,$$

so

a) Two L. I. solution

one : $X_0(x)T_0(t) = e^{-\left[\frac{\pi}{2L}\right]^2 t} \cos\frac{\pi x}{L}$

second : $X_1(x)\,T_1(t) = e^{-\left(\frac{3\pi}{2L}\right)^2 t} \cos\frac{3\pi x}{L}.$

b). step 2 :

$$u(x,t) = \sum_{n=0}^{\infty} C_n\, e^{-\left[\frac{(2n+1)\pi}{2L}\right]^2 t} \cos\frac{(2n+1)\pi x}{2L}.$$

$$u(x,0) = \cos\frac{3\pi x}{2L}$$

$$\Rightarrow \sum_{n=0}^{\infty} C_n \cos\frac{(2n+1)\pi x}{2L} = \cos\frac{3\pi x}{2L}$$

So $C_0 = 0, \quad C_1 = 1, \quad C_2 = 0, \quad C_3 = C_4 = \cdots = 0$

Thus $u(x,t) = e^{-\left(\frac{3\pi}{2L}\right)^2 t} \cos\frac{3\pi x}{2L}$.