MATH 1823 Honors Calculus I
Rolle's Theorem, MVT and friends

Due in class on Friday, December 1, 2000

Generalized Mean Value Theorem. Here are the hypotheses:

- $f(x)$ and $g(x)$ are continuous on $[a, b]$
- $f(x)$ and $g(x)$ are differentiable on (a, b)
- $g'(x) \neq 0$ for all x in (a, b)

and here's the conclusion:

- there exists a number c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Q1]... Say what is wrong with the following *proof*.

By the MVT (we are told that f satisfies the correct hypotheses of the MVT above) there is a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Also by the MVT (we are told that g satisfies the correct hypotheses of the MVT above) there is a point c in (a, b) such that

$$g'(c) = \frac{g(b) - g(a)}{b - a}.$$

Now, we are also told that $g'(c) \neq 0$, so we can divide the first equation by the second to get

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{b - a} \frac{b - a}{g(b) - g(a)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Done! [...or are we?...]

Q2]... This should help you answer question 1 — if you haven't spotted any obvious errors in the *proof* above, then you should take a look at it again, after answering this question!

Let $f(x) = x^2$ and $g(x) = x^3$ be functions on the interval $[0, 1]$.

- Find the number(s) c guaranteed by the MVT for the function f on $[0, 1]$.
- Find the number(s) c guaranteed by the MVT for the function g on $[0, 1]$.
- Find the number(s) c guaranteed by the GMVT for the functions f and g on $[0, 1]$.

So you've seen what's wrong with the first attempt at a proof. Here is the outline of a proof which is similar in spirit to the proof of the MVT that we met in class.

Q3]... Consider the function (h for *horrible*)

$$h(x) \; = \; [f(b) - f(a)][g(x) - g(a)] - [g(b) - g(a)][f(x) - f(a)]$$

Compute/verify the following (as appropriate)

- h is continuous on $[a, b]$
- h is differentiable on (a, b)
- Compute $h'(x)$
- Compute $h(a)$
- Compute $h(b)$

What theorem can we appeal to now? Complete the proof of the GMVT.

Now we give a cool application of the GMVT. You will use this application more often than you care to admit in a typical calculus sequence! Again, it has a dead guy's name attached to it: *l'Hospital's Rule.*

The version we shall establish is for a limit as $x \to a^+$. All other forms of l'Hospital's Rule can be derived from this case. These include limits where $x \to a^-$, and $x \to a$, and where $a = \pm\infty$ and where $f(x)$ and $g(x)$ are both tending to $\pm\infty$ instead of to 0. We wont derive these here!!

Suppose

- f and g are differentiable on $[a, b]$
- $g'(x) \neq 0$ in (a, b)
- $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$
- $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$

Then

- $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$ too.

This is often written out in shorthand (minus the hypotheses) as

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} \; = \; \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

Q4]... Prove the version of l'Hospital's Rule above. Note that for any x in (a, b) we can say that there is a number c between a and x such that

$$\frac{f(x) - f(a)}{g(x) - g(a)} \; = \; \frac{f'(c)}{g'(c)}$$

.

- Why can we say this?
- Also, how do we know for sure that the denominator on the left hand side is not zero?
- But $f(a) = 0 = g(a)$ (why?) and so the equation above simplifies down to what?
- what happens to c as $x \to a^+$?
- Complete the proof of l'Hospital's Rule.

Q5]... Use l'Hospital's Rule ($x \to 0$ case instead of $x \to 0^+$) to compute the following limit

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2}$$

Q6]... We have seen in class and the homeworks (Q34 pge 135) that the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has a derivative of 0 at $x = 0$. **You do not have to recheck this.**

Use the usual differentiation rules (product, chain, power etc) to find the derivative of $f(x)$ when $x \neq 0$. Conclude that

$$f'(x) = \begin{cases} -\cos \frac{1}{x} + 2x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Note that this function is **not continuous** at $x = 0$.
So, the derivative of a function need not itself be a continuous function.

Q7]... A natural question to ask (if you are an evil torturing math prof for instance) is *which* functions can be derivatives of other functions. We shall see in Calculus II that every continuous function is the derivative of some function. We have seen in the example above, that certain non-continuous functions can be derivatives too. What about the function $g(x)$ below?

$$g(x) = \begin{cases} -1 & \text{when } x < 0 \\ 1 & \text{when } x \geq 0 \end{cases}$$

Let is suppose $g(x)$ is the derivative $f'(x)$ of some function f. Show that this assumption leads us into a contradiction by using the MVT three times: once for $\frac{f(1)-f(0)}{1-0}$, once for $\frac{f(0)-f(-1)}{0-(-1)}$, and once for $\frac{f(1)-f(-1)}{1-(-1)}$.

We finish off by showing that derivatives must satisfy an intermediate value property (akin to the IVT for continuous functions). This is called the Darboux property of derivatives. This will rule out functions like $g(x)$ above which have jump discontinuities.

Darboux Property: Suppose f is differentiable at every point of $[a, b]$. Then f' takes on every value between $f'(a)$ and $f'(b)$.

Q8]... Work through this proof.

Suppose that $f'(a) < f'(b)$ (the case where $f'(a) > f'(b)$ is handled similarly), and that γ lies strictly between $f'(a)$ and $f'(b)$.

Consider the function

$$g(x) = f(x) - \gamma x$$

- Show that g is continuous on $[a, b]$.
- Use the extreme value theorem (pge 225) to conclude that g attains a minimum somewhere in $[a, b]$.

- Show that this minimum cannot occur at the endpoint a. Hint: show that $g'(a) < 0$. What does this tell you?
- Show that this minimum cannot occur at the endpoint b. Hint show that $g'(b) > 0$. What does this tell you?
- Thus there is a point c strictly between a and b where g attains its minimum. What does Fermat tell us about $g'(c)$?
- Finish the proof of the Darboux property!
