Q1]. . . [10 points] Evaluate the following two limits, showing all your work.

$$\lim_{t\to 1} \frac{t^2-1}{t^2-2t+1}$$

$$\lim_{t\to 1}\left(\frac{t^2-1}{t^2-2t+1}\right) = \lim_{t\to 1}\left(\frac{\cancel{(t-1)}(t+1)}{\cancel{(t-1)}(t-1)}\right) = \lim_{t\to 1}\left(\frac{t+1}{t-1}\right)$$

as $t\to 1$, numerator $\to 2$

as $t\to 1^-$ denominator $\to 0^-$) $\Rightarrow$ fraction $\to -\infty$

as $t\to 1^+$ denominator $\to 0^+$) $\Longrightarrow$ fraction $\to +\infty$

Therefore Limit DNE (Does not exist).

$$\lim_{x\to 8} \frac{x^{2/3}-4}{x^{1/3}-2}$$

$$\text{Lim} = \lim_{x\to 8}\left(\frac{(x^{1/3})^2-4}{(x^{1/3})-2}\right) = \lim_{x\to 8}\left(\frac{\cancel{(x^{1/3}-2)}(x^{1/3}+2)}{\cancel{(x^{1/3}-2)}}\right)$$

$$= \lim_{x\to 8}\left(x^{1/3}+2\right)$$

$$= 8^{1/3}+2 = 2+2 = \boxed{4}$$

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Q2]... **[10 points]** Evaluate the following limits, showing all your work.

$$\lim_{x\to\infty}(\sqrt{x^2+2x}-\sqrt{x^2-2x})$$

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)}{1}\frac{\left(\sqrt{x^2+2x}+\sqrt{x^2-2x}\right)}{\left(\sqrt{x^2+2x}+\sqrt{x^2-2x}\right)}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}+2x-(\cancel{x^2}-2x)}{\sqrt{x^2+2x}+\sqrt{x^2-2x}} = \lim_{x\to\infty} \frac{4x}{\left(\sqrt{x^2+2x}+\sqrt{x^2-2x}\right)}$$

$$= \lim_{x\to\infty} \left(\frac{4}{\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{2}{x}}}\right) = \frac{4}{\sqrt{1}+\sqrt{1}} = \frac{4}{2} = \boxed{2}$$

$$\lim_{t\to 2+}\frac{t}{(2-t)^3}$$

$$\lim_{t\to 2^+} \frac{t}{(2-t)^3} = \boxed{-\infty}$$ since

as $t \to 2+$ $\quad (2-t) \to 0^-$

& $(2-t)^3 \to 0^-$

also $t \to 2$

$\Rightarrow$ fraction $\to -\infty$

Q3]... [10 points] Find the **equation of the tangent line** to the graph of the function $y = (x-1)^{1/3}$ at the point $(2, 1)$. Show all your work carefully. Note that the $1/3$ is an exponent (and so denotes a cube root). You are **only** to work with ideas and techniques from chapters 1 and 2 of the book (do not use quick methods or special rules).

Equation of line $\boxed{(y-y_1) = m(x-x_1)}$ — Point: (x_1, y_1), slope: m

1st need slope

$$= \lim_{h\to 0}\left(\frac{(2+h-1)^{1/3} - (2-1)^{1/3}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{(h+1)^{1/3} - 1}{h}\right)$$

Recall $\boxed{(A-B)(A^2+AB+B^2) = A^3 - B^3}$

$$= \lim_{h\to 0} \frac{\left((h+1)^{1/3} - 1\right)\left((h+1)^{2/3} + (h+1)^{1/3} + 1\right)}{h\left((h+1)^{2/3} + (h+1)^{1/3} + 1\right)}$$

$$= \lim_{h\to 0} \frac{\left((h+1)^{1/3}\right)^3 - 1^3}{h\left((h+1)^{2/3} + (h+1)^{1/3} + 1\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left((h+1)^{2/3} + (h+1)^{1/3} + 1\right)}$$

$$= \frac{1}{1^{2/3} + 1^{1/3} + 1} = \boxed{\frac{1}{3}}$$

Now, point $(2,1)$ so equation is:

$$\boxed{(y-1) = \frac{1}{3}(x-2)}$$

Q4]... [10 points] Find the value of the constant c which makes the function f below continuous. Show how you obtained your answer.

$$f(x) = \begin{cases} x^2 + c & \text{if} \quad x \geq 1 \\ 7x - 1 & \text{if} \quad x < 1 \end{cases}$$

continuous at $x = 1$? Justify your answer.

$f(1) = (1)^2 + c = \boxed{1+c}$ --- by def^n of $f(x)$ above

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (7x-1)$ --- by def^n of $f(x)$ above

$= 7(1) - 1 = \boxed{6}$

Those should be equal for continuity

$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$
$= \lim_{x \to a} f(x) = f(a)$

Therefore, $1 + c = 6 \Rightarrow \boxed{c = 5}$

Is the function

$$g(x) = \begin{cases} x \sin(\frac{1}{x}) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

continuous at $x = 0$? Justify your answer.

Yes ... since $-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$

$-|x| \to 0$ as $x \to 0$; $|x| \to 0$ as $x \to 0$

& so $\lim_{x \to 0} \left(x \sin\left(\frac{1}{x}\right)\right) = 0$ by Squeeze Th^m.

But this agrees with $g(0)$.

Thus $\lim_{x \to 0} g(x) = 0 = g(0)$ & so $g(x)$ is indeed continuous at 0.

Q5]... [10 points] Write down an expression (no proof necessary) for the sine of the sum of two angles in terms of the sines and cosines of the two angles.

$\sin(x+h) =$ $\sin(x)\cos(h) + \cos(x)\sin(h)$

Show how the expression above is used in showing that the sine function is continuous at an arbitrary point x.

We want to show

$$\lim_{t \to x} \sin(t) = \sin(x)$$

OR which is the same thing (write $t = x+h$

Note: $h = t - x \to x - x = 0$ as $t \to x$)

as

$$\lim_{h \to 0} \left(\sin(x+h) \right) = \sin(x) \quad \text{———} \ (*)$$

$$\text{Left side of } (*) = \lim_{h \to 0} \left(\sin(x)\cos(h) + \cos(x)\sin(h) \right)$$

$$= \sin(x) \underbrace{\lim_{h \to 0} (\cos(h))}_{1} + \cos(x) \underbrace{\lim_{h \to 0} (\sin(h))}_{0}$$ ... Limit Laws,

1 proven in class

0 proven (geometrically) in class

$$= \sin(x) \cdot 1 + \cos(x) \cdot 0$$

$$= \sin(x)$$ & we're done!