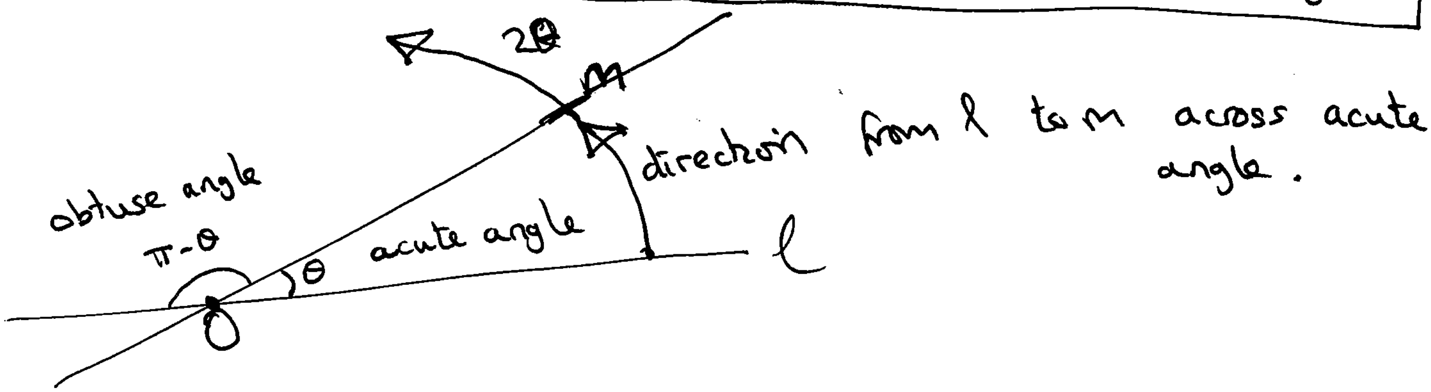
Cool Fact: Let l denote 'reflection in the line l' and m denote 'reflection in the line m'.

If lines l, m meet in a point O, and make an angle of θ, then:

$$\boxed{m \circ l = \text{Rotation about } O \text{, through an angle of } 2\theta \text{, in direction from } l \text{ to } m \text{ (across acute angle).}}$$



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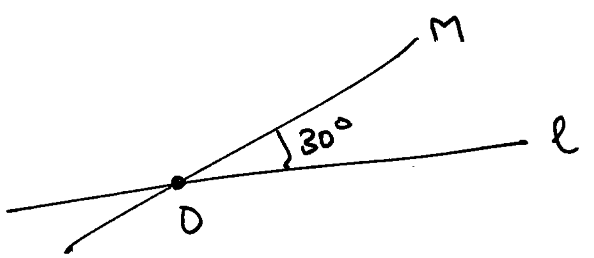
eg.



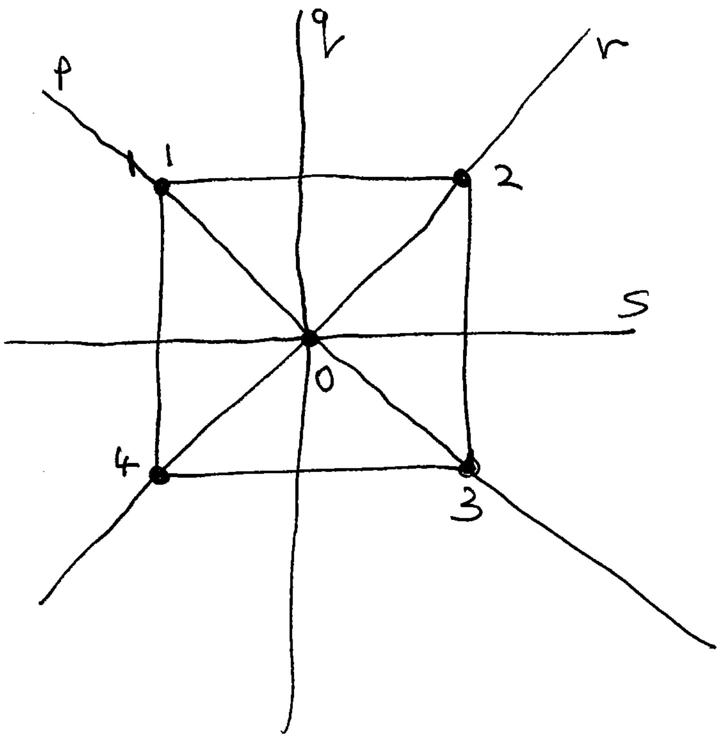
$m \circ l$ = counter clockwise Rotation about O through $2(30) = 60°$.

$l \circ m$ = clockwise rotation about O through $2(30) = 60°$.

Note: $(l \circ m) \circ (m \circ l) = l \circ \underbrace{\cancel{m^2}}_{=\mathbb{1}} \circ l$

$= \underbrace{\cancel{l^2}}_{=\mathbb{1}} = \mathbb{1}$

Makes sense, since clockwise + counterclockwise rotations cancel out!



$p \longleftrightarrow (24)$

$q \longleftrightarrow (12)(34)$

$r \longleftrightarrow (13)$

$s \longleftrightarrow (14)(23)$

$R \longleftrightarrow (1432)$

$R^2 \longleftrightarrow (13)(24)$

$R^3 \longleftrightarrow (1234)$

$\mathbb{1} = R^4 \longleftrightarrow \mathbb{1}$

$R =$ counter clock wise Rotation about O through $\pi/2$.

$R^2 = R \circ R$ $\qquad$ $R^3 = R \circ R \circ R$ $\qquad$ $R^4 = R \circ R \circ R \circ R = \mathbb{1}$.

composition symbol → $\circ$	$\mathbb{1}$	R	R^2	R^3	p	q	r	s
$\mathbb{1}$	$\mathbb{1}$	R	R^2	R^3	p	q	r	s
R	R	R^2	R^3	$\mathbb{1}$	s	p	q	r
R^2	R^2	R^3	$\mathbb{1}$	R	r	s	p	q
R^3	R^3	$\mathbb{1}$	R	R^2	q	r	s	p
p	p	q	r	s	$\mathbb{1}$	R	R^2	R^3
q	q	r	s	p	R^3	$\mathbb{1}$	R	R^2
r	r	s	p	q	R^2	R^3	$\mathbb{1}$	R
s	s	p	q	r	R	R^2	R^3	$\mathbb{1}$

IDEA

- TOP LEFT SQUARE IS EASY!!
- Bottom RIGHT SQUARE is OK!! (Just use "cool fact" over & over again.)
- For remaining squares use algebra!

eg

$$\underline{R} \circ p = (s \circ \underline{p}) \circ p = s \circ (p \circ p) = s \circ \mathbb{1} = s$$

Read ~~R~~ -- from bottom right square!

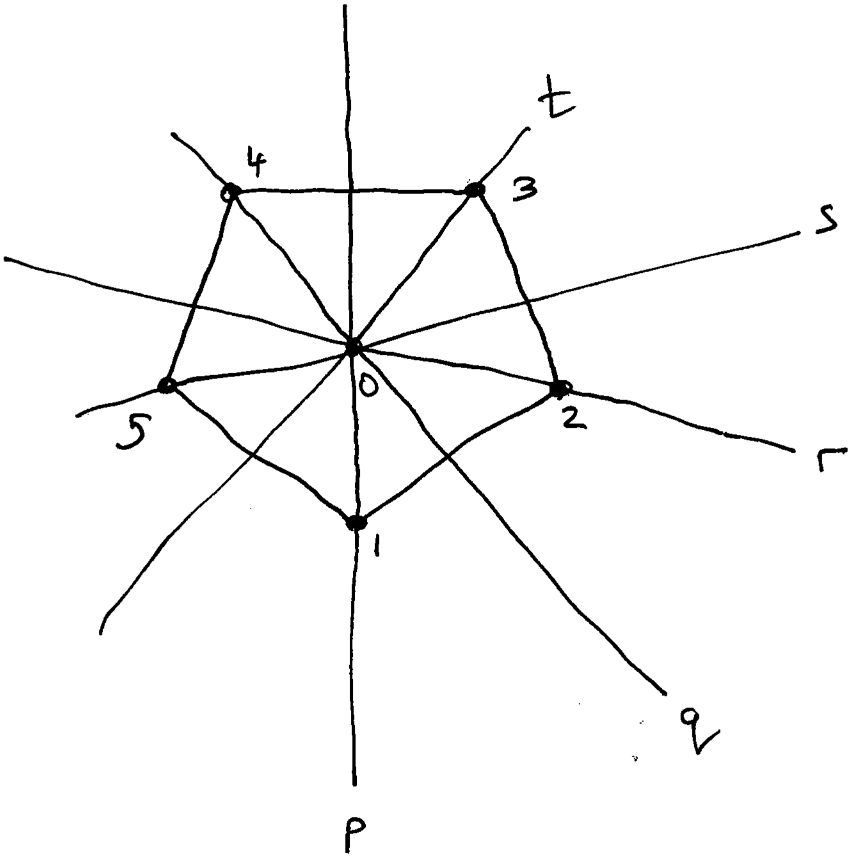
and eg

$$p \circ \underline{R} = p \circ (p \circ \underline{q}) = (p \circ p) \circ q = \mathbb{1} \circ q = q.$$

Read from bottom right square!

There are very fast ways of reading off & filling in values — via rows/colums of original table.

Think about it!!



$p \leftrightarrow (25)(43)$

$q \leftrightarrow (12)(35)$

$r \leftrightarrow (45)(13)$

$s \leftrightarrow (14)(23)$

$t \leftrightarrow (15)(24)$

$R \leftrightarrow (12345)$

$R^2 \leftrightarrow (13524)$

$R^3 \leftrightarrow (14253)$

$R^4 \leftrightarrow (15432)$

$\mathbb{1} = R^5 \leftrightarrow \mathbb{1}$

$R =$ Rotation counter clockwise about O through $\frac{2\pi}{5}$

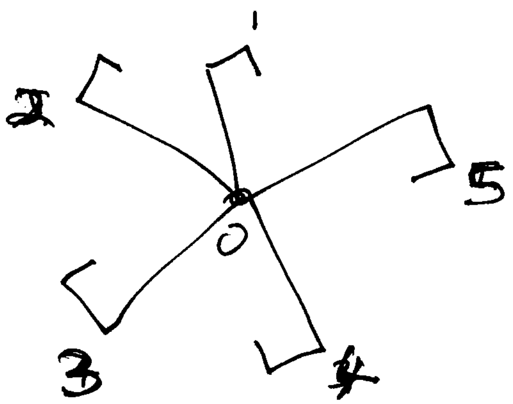
$R^2 = R \circ R, \quad R^3 = R \circ R^2, \quad R^4 = R^3 \circ R, \quad R^5 = R^4 \circ R = \mathbb{1}.$

There are 5 Reflections in lines : p, q, r, s, t
and 5 Rotations about O : $R, R^2, R^3, R^4, R^5 = \mathbb{1}$.

Comp table.

$\circ$	$\mathbb{1}$	R	R^2	R^3	R^4	p	q	r	s	t
$\mathbb{1}$	$\mathbb{1}$	R	R^2	R^3	R^4	p	q	r	s	t
R	R	R^2	R^3	R^4	$\mathbb{1}$	q	r	s	t	p
R^2	R^2	R^3	R^4	$\mathbb{1}$	R	r	s	t	p	q
R^3	R^3	R^4	$\mathbb{1}$	R	R^2	s	t	p	q	r
R^4	R^4	$\mathbb{1}$	R	R^2	R^3	t	p	q	r	s
p	p	t	s	r	q	$\mathbb{1}$	R^4	R^3	R^2	R
q	q	p	t	s	r	R	$\mathbb{1}$	R^4	R^3	R^2
r	r	q	p	t	s	R^2	R	$\mathbb{1}$	R^4	R^3
s	s	r	q	p	t	R^3	R^2	R	$\mathbb{1}$	R^4
t	t	s	r	q	p	R^4	R^3	R^2	R	$\mathbb{1}$

Some patterns should emerge as you fill in the squares of



Answer: No reflection symmetries.
Just Rotations about O, through

$$\frac{2\pi}{5},\ 4\pi/5,\ \frac{6\pi}{5},\ \frac{8\pi}{5},\ \frac{10\pi}{5} \leftrightarrow \mathbb{1}.$$

$$R \leftrightarrow (12345)$$
$$R^2 \leftrightarrow (13524)$$
$$R^3 \leftrightarrow (14253)$$
$$R^4 \leftrightarrow (15432)$$
$$\mathbb{1} = R^5 \leftrightarrow \mathbb{1}$$

They form a (~~close~~) subset of perm($\{1, \_, 5\}$) which are closed under composition

$$\underbrace{\{1, R, R^2, R^3, R^4\}}_{\text{has 5 elements}} \subseteq \underbrace{\left\{\begin{matrix}\mathbb{1}, R, R^2, R^3, R^4 \\ p, q, r, s, t\end{matrix}\right\}}_{\text{Has 10 elements}} \subseteq \underbrace{\mathrm{Perm}(\{1, 2, 3, 4, 5\})}_{\text{Has 120 elements}}$$

<u>Note</u>: 5 divides 10, & 10 divides 120.