Homework-4 Solutions

Section 2.3 (90-91)

37) Prove that $\lim_{x\to 0} x^4 \cos\frac{2}{x} = 0$

$$-1 \le \cos\left(\frac{2}{x}\right) \le 1$$

$$\Rightarrow -x^4 \le x^4\cos\left(\frac{2}{x}\right) \le x^4 \quad \text{—}(1)$$

as $x \to 0$, $x^4 \to 0$ and $-x^4 \to 0$

By using Squeeze theorem in (1) we get,

as $x \to 0$, $x^4\cos\left(\frac{2}{x}\right) \to 0$

(or) $\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$

38) $\lim_{x\to 0^+} \sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right] = 0$

$$-1 \le \sin\left(\frac{2\pi}{x}\right) \le 1$$

$$\Rightarrow 0 \le \sin^2\left(\frac{2\pi}{x}\right) \le 1$$

$$\Rightarrow 1 \le 1+\sin^2\left(\frac{2\pi}{x}\right) \le 2$$

$$\Rightarrow \sqrt{x} \le \sqrt{x}\left(1+\sin^2\left(\frac{2\pi}{x}\right)\right) \le 2\sqrt{x} \quad \text{—}(1)$$

as $x \to 0$, $\sqrt{x} \to 0$ and $2\sqrt{x} \to 0$

By using Squeeze theorem in (1), we get,

as $x \to 0$, $\sqrt{x}\left(1+\sin^2\left(\frac{2\pi}{x}\right)\right) \to 0$

(or) $\lim_{x\to 0^+} \sqrt{x}\left(1+\sin^2\left(\frac{2\pi}{x}\right)\right) \to 0$

40) $\lim\limits_{x \to -4^-} \dfrac{|x+4|}{(x+4)}$

Since we are evaluating the LHL at 4, we approach -4 from the left of -4

i.e. $x < -4$ (or) $(x+4) < 0$

$$|x+4| = \begin{cases} x+4 & x+4 \geq 0 \\ -(x+4) & x+4 < 0 \end{cases}$$

Since $(x+4) < 0$,

$$\lim_{x \to -4^-} \frac{|x+4|}{(x+4)} = \lim_{x \to -4^-} \frac{-(x+4)}{(x+4)} = \lim_{x \to -4^-} -1 = -1$$

(ie) $\lim\limits_{x \to -4^-} \dfrac{|x+4|}{(x+4)} = -1$

42) $\lim\limits_{x \to \frac{3}{2}} \dfrac{2x^2 - 3x}{|2x-3|}$

Consider

$$\lim_{x \to \frac{3}{2}^-} \frac{2x^2-3x}{|2x-3|}$$

$x < \frac{3}{2} \Rightarrow x - \frac{3}{2} < 0$

$$|2x-3| = \begin{cases} 2x-3 & x \geq 3/2 \quad (2x-3 \geq 0) \\ -(2x+3) & x < 3/2 \quad (2x-3 < 0) \end{cases}$$

As $x < \frac{3}{2}$, $\lim\limits_{x \to \frac{3}{2}^-} \dfrac{2x^2-3x}{|2x-3|} = \lim\limits_{x \to \frac{3}{2}^-} \dfrac{x\cancel{(2x-3)}}{-\cancel{(2x-3)}} = -\dfrac{3}{2}$

$$\lim_{x \to \frac{3}{2}^+} \frac{2x^2-3x}{|2x-3|} = \lim_{x \to \frac{3}{2}^+} \frac{x(2x-3)}{2x-3} = \frac{3}{2}$$

$$\lim_{x \to \frac{3}{2}^+} \frac{2x^2-3x}{|2x-3|} = \frac{3}{2} \quad \text{and} \quad \lim_{x \to \frac{3}{2}^-} \frac{2x^2-3x}{|2x-3|} = -\frac{3}{2}$$

LHL and RHL dont match at $3/2$

$$\therefore \lim_{x \to \frac{3}{2}} \frac{2x^2-3x}{|2x-3|} \quad \text{D.N.E}$$

44) $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$

As we are evaluating the RHL at 0

$x > 0 \Rightarrow |x| = x$

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right) = \lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{x}\right) = \lim_{x \to 0^+} 0 = 0$$

(ie) $\lim_{x \to 0^+} \frac{1}{x} - \frac{1}{|x|} = 0$

58) $$\lim_{x \to 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1}$$

$$= \lim_{x \to 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1} \times \frac{\sqrt{3-x} + 1}{\sqrt{3-x} + 1} \times \frac{\sqrt{6-x} + 2}{\sqrt{6-x} + 2}$$

$$= \lim_{x \to 2} \frac{(\sqrt{6-x} - 2)(\sqrt{6-x} + 2)(\sqrt{3-x} + 1)}{(\sqrt{3-x} - 1)(\sqrt{3-x} + 1)(\sqrt{6-x} + 2)}$$

$$= \lim_{x \to 2} \frac{(6-x-4)(\sqrt{3-x} + 1)}{(3-x-1)(\sqrt{6-x} + 2)}$$

$$= \lim_{x \to 2} \frac{\cancel{2-x}}{\cancel{2-x}} \times \frac{\sqrt{3-x} + 1}{\sqrt{6-x} + 2}$$

$$= \lim_{x \to 2} \frac{\sqrt{3-x} + 1}{\sqrt{6-x} + 2} = \frac{\sqrt{1} + 1}{\sqrt{4} + 2} = \frac{2}{4} = \frac{1}{2}$$

(ie) $$\lim_{x \to 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1} = \frac{1}{2}$$

16) Section 2.5 (111 – 112)

16) $f(x) = \begin{cases} \frac{1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$ $a = 1$

$f(1) = 2$

$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{1}{x-1}$ D.N.E

So $\lim_{x \to 1} f(x) \neq f(1)$

$\therefore$ f is discontinuous at $a = 1$

18) $f(x) = \begin{cases} \frac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$ $a = 1$

$f(1) = 1$

$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = \frac{1}{2}$

$\lim_{x \to 1} f(x) = \frac{1}{2} \neq f(1)$

$\therefore$ f is discontinuous at $a = 1$

20) $f(x) = \begin{cases} 1+x^2 & \text{if } x<1 \\ 4-x & \text{if } x \geq 1 \end{cases}$ $\quad a=1$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 1+x^2 = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4-x = 3$$

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

$\therefore$ f is discontinuous at $a=1$

24) $h(x) = \dfrac{\sin x}{x+1} = \dfrac{f(x)}{g(x)}$

$f(x) = \sin x$ is a trigonometric function that is continuous at every point in its domain $\mathbb{R}$ [Th 7]

$g(x) = x+1$ is a polynomial function. So its continuous on its domain $\mathbb{R}$ [Th. 7]

$h(x) = \dfrac{f(x)}{g(x)}$ is continuous at all points x such that $g(x) \neq 0$ (as $h(x)$ is not defined when $g(x) = 0$) [Th. 4]

$g(x) = 0 \Rightarrow x+1 = 0 \Rightarrow x = -1$

So $h(x) = \dfrac{f(x)}{g(x)}$ is defined and continuous at all points x such that $x \neq -1$.

Domain of $h = \{x \in \mathbb{R} : x \neq 1\} = (-\infty, -1) \cup (-1, \infty)$

26) $h(x) = \tan 2x$

$$\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{f(x)}{g(x)}$$

$f(x) = \sin 2x$ is defined and continuous at all points in its domain $\mathbb{R}$.

$g(x) = \cos 2x$ is defined and continuous at all points in its domain $\mathbb{R}$.

$h(x) = \dfrac{f(x)}{g(x)}$ is defined and continuous at all points in $\mathbb{R}$ such that $g(x) \neq 0$

$$g(x) = 0 \Rightarrow \cos 2x = 0 \Rightarrow 2x = n\pi + \frac{\pi}{2}$$

$$(\text{or}) \quad x = \frac{n\pi}{2} + \frac{\pi}{4}$$

$h(x) = \tan 2x$ is defined at all points x such that $x \neq \dfrac{n\pi}{2} + \dfrac{\pi}{4}$

Domain of $h = \{x \in \mathbb{R} : x \neq \frac{n\pi}{2} + \frac{\pi}{4},\ n \in \mathbb{Z}\}$

43) $f(x) = x^3 - x^2 + x$

Consider the function $h(x) = x^3 - x^2 + x - 10$
$[= f(x) - 10]$

h is a polynomial function. Its continuous at every point in $\mathbb{R}$.

In particular, its continuous at every point in $[2, 3]$

$h(2) = 2^3 - 2^2 + 2 - 10 = -4 < 0$

$h(3) = 3^3 - 3^2 + 3 - 10 = 11 > 0$

$h(2) > 0$, $h(3) > 0$ and h is continuous in $[2,3]$ $\Rightarrow$ By IVT there exists $c \in (2,3)$ such that $h(c) = 0$

(ie) $f(c) - 10 = 0$

(or) $f(c) = 10$

So there is a number c such that

$f(c) = 10$

44) Consider the function $f(x) = x^2 - 2$

f is a polynomial. f is continuous at every point in $\mathbb{R}$.

In particular, f is continuous in $[1, 2]$.

$f(1) = 1^2 - 2 = -1 < 0$

$f(2) = 2^2 - 2 = 2 > 0$

$f(1) < 0$, $f(2) > 0$ and f is continuous in $[1, 2]$ $\Rightarrow$ By IVT $\exists$ $c \in (1, 2)$ s.t

$f(c) = 0$

(ie) $\exists$ $c \in (1, 2)$ st $c^2 - 2 = 0$ (or) $c^2 = 2$.

x ———— x