Solutions (Homework-9)

6) Particle moves along curve $y = \sqrt{1+x^3}$

$y = \sqrt{1+x^3} \Rightarrow \frac{dy}{dt} = \frac{1}{2\sqrt{1+x^3}} \times 3x^2 \frac{dx}{dt} = \frac{3x^2}{2\sqrt{1+x^3}} \frac{dx}{dt}$

Given $\frac{dy}{dt} = 4\text{cm/s}$,

As it reaches $(2,3)$, $x=2$, $y=3$

$\frac{dy}{dt} = \frac{3x^2}{2\sqrt{1+x^3}} \frac{dx}{dt} \Rightarrow 4 = \frac{3 \cdot 4}{2\sqrt{1+8}} \frac{dx}{dt}$

$\boxed{\frac{dx}{dt} = 2\text{cm/s}}$

8) A snowball is a sphere

Surface area$(S) = 4\pi r^2$

Diameter$(di) = 2r$

$\Rightarrow S = 4\pi r^2 = \pi (2r)^2 = \pi di^2$

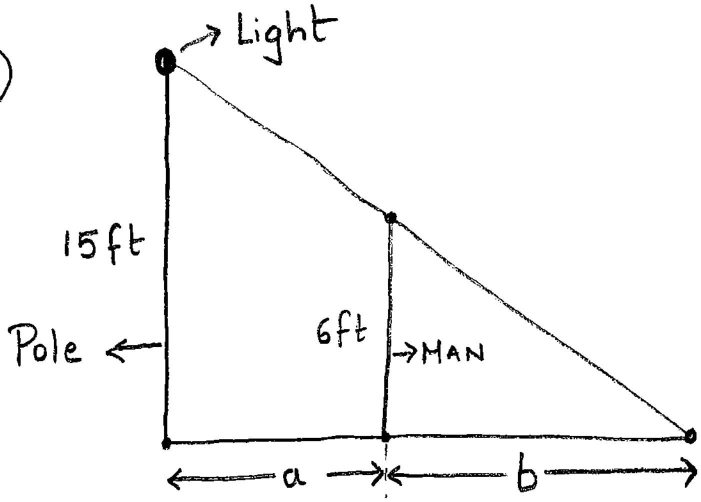
Given $\frac{ds}{dt} = -1\text{cm}^2/\text{min}$ (since it decreases)

$\frac{ds}{dt} = 2\pi di \left(\frac{d(di)}{dt}\right)$

When $di = 10\text{cm}$

$-1 = 2\pi(10)\left(\frac{d(di)}{dt}\right) \Rightarrow \boxed{\frac{d(di)}{dt} = -\frac{1}{20\pi}\text{cm/s}}$

9)



Let 'a' be the distance of the pole from the man

Let 'b' be the distance of the man from the tip of his shadow

We want to find $\frac{d}{dt}(a+b)$, when $a = 40ft$

By similar triangles.

$$\frac{6}{15} = \frac{b}{a+b} \Rightarrow 6a = 9b \Rightarrow b = \frac{2}{3}a$$

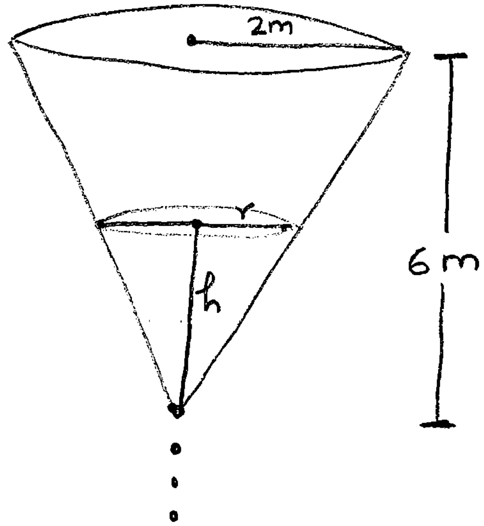
$$\frac{d}{dt}(a+b) = \frac{d}{dt}\left(a + \frac{2}{3}a\right) = \frac{d}{dt}\left(\frac{5}{3}a\right) = \frac{5}{3}\frac{da}{dt}$$

$$\frac{da}{dt} = \text{speed of man} = 5ft/sec$$

$$\Rightarrow \frac{d}{dt}(a+b) = \frac{5}{3}.5 = \frac{25}{3} ft/sec$$

$$\boxed{\frac{d}{dt}(a+b) = \frac{25}{3} ft/sec}$$

16) Already done in class.

19) 

height = 6m = 600 cm

radius = 2m = 200 cm

Water is being pumped in at $k\ cm^3/min$ (say)

Water is leaking at $10000\ cm^3/min$

$\Rightarrow$ Rate of change of volume $\left(\frac{dv}{dt}\right) = k - 10000\ cm^3/min$

By similar triangles,

$$\frac{h}{6} = \frac{r}{2} \Rightarrow r = \frac{h}{3}$$

$$\text{Volume } (V) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2(h) = \frac{\pi h^3}{27}$$

$$\frac{dv}{dt} = \frac{\pi h^2}{9}\left(\frac{dh}{dt}\right)$$

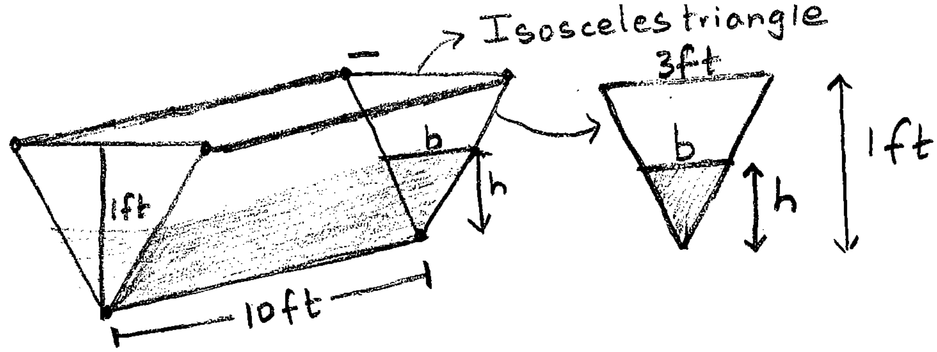
Given $\frac{dh}{dt} = 20\ cm/min$

When $h = 2m = 200\ cm$

$$\frac{dv}{dt} = \frac{\pi h^2}{9}\left(\frac{dh}{dt}\right) \Rightarrow k - 10000 = \frac{\pi (600)^2}{9}(20)$$

$$\Rightarrow \boxed{k = 10000 + 800000\pi\ cm^3/min}$$

20)



By similar triangles $\frac{b}{3} = \frac{h}{1} \Rightarrow b = 3h$

Volume of water $(V) = \left(\frac{1}{2} \cdot b \cdot h\right) \times$ (length of trough)

$= \frac{1}{2} bh \cdot 10 = 5bh = 15h^2$

We want to find $\frac{dh}{dt}$, when $h = \frac{1}{2}$ ft

$\frac{dv}{dt} = 30h\left(\frac{dh}{dt}\right)$

Given, $\frac{dh}{dt} = 20$ cm/min.

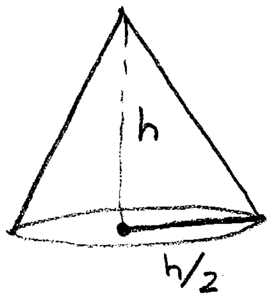
When $h = \frac{1}{2}$ ft

$\frac{dv}{dt} = 30h\left(\frac{dh}{dt}\right) \Rightarrow 12 = 30 \cdot \frac{1}{2} \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4}{5}$ ft/min

Rate at which water level is rising

$= \boxed{\frac{dh}{dt} = \frac{4}{5} \text{ ft/min}}$

23)



diameter of base (d) = height (h)

$\Rightarrow 2r = h$ (or) $r = h/2$

Volume of pile (V) $= \frac{1}{3}\pi r^2 h = \frac{1}{3} \cdot \pi \cdot \frac{h^2}{4} h$

$= \frac{1}{12}\pi h^3$

Rate of change of volume $\left(\frac{dv}{dt}\right) = \frac{\pi h^2}{4}\left(\frac{dh}{dt}\right)$

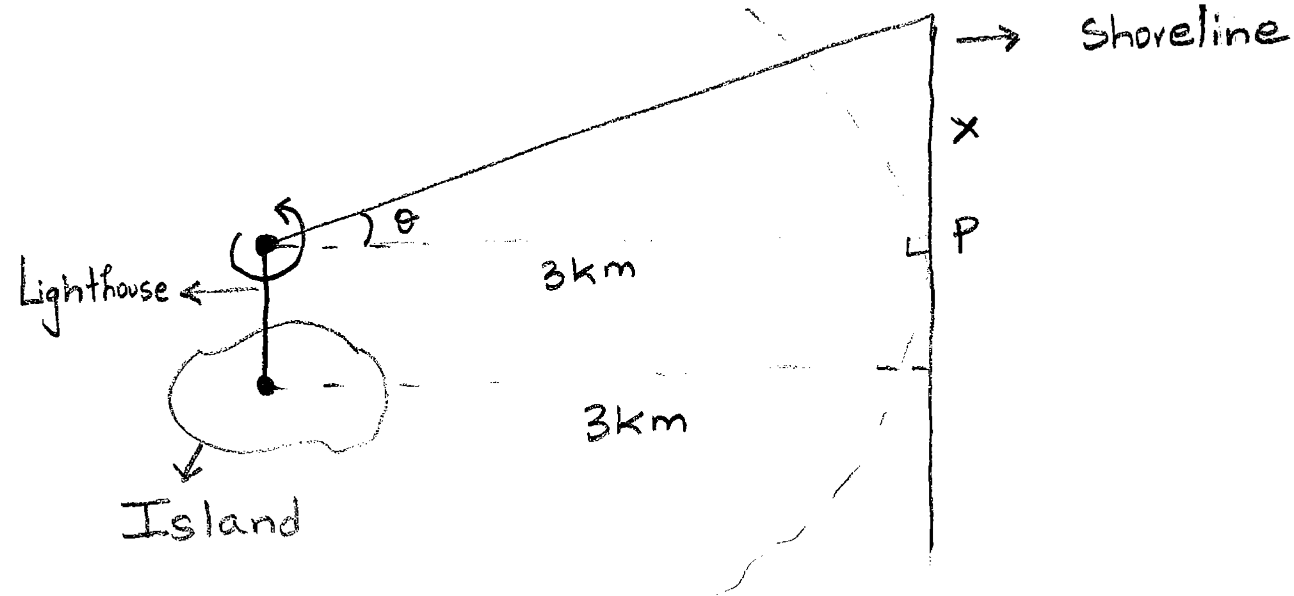
Given : $\frac{dv}{dt} = 30\ ft^3/min$

We want to find $\frac{dh}{dt}$ when $h = 10ft$

$\frac{dv}{dt} = \frac{\pi h^2}{4}\left(\frac{dh}{dt}\right) \Rightarrow 30 = \frac{\pi(100)}{4}\left(\frac{dh}{dt}\right)$

$\Rightarrow \boxed{\frac{dh}{dt} = \frac{6}{5\pi}\ ft/min}$

34)



Speed of rotation (of light) $\left(\frac{d\theta}{dt}\right) = 4$ rev/min

$= 8\pi$ rad/min

x = distance of light from P

$\tan\theta = \frac{x}{3} \Rightarrow x = 3\tan\theta$

We want to find $\frac{dx}{dt}$ when $x = 1$

$$\frac{dx}{dt} = 3\sec^2\theta \left(\frac{d\theta}{dt}\right)$$

When $x = 1$, $\tan\theta = \frac{1}{3}$, $\sec^2\theta = 1 + \tan^2\theta$

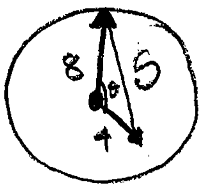
$= 1 + \frac{1}{9} = \frac{10}{9}$

$$\frac{dx}{dt} = 3\left(\frac{10}{9}\right) 8\pi = \frac{80\pi}{3} \text{ km/min}$$

$$\boxed{\frac{dx}{dt} = \frac{80\pi}{3} \text{ km/min}}$$

38) The hour hand goes around every 12 hours

$\Rightarrow$ speed of hour hand $= \frac{2\pi}{12} = \frac{\pi}{6}$ rad/h



On the other hand, the minute hand goes around every 1 hour

$\Rightarrow$ speed of minute hand $= \frac{2\pi}{1} = 2\pi$ rad/h

If θ is the angle between them.

$\frac{d\theta}{dt} = -\left(2\pi - \frac{\pi}{6}\right) = -\frac{11\pi}{6}$ rad/h (negative as θ decreases a the minute hand moves towards hour hand)

Let s = distance between them

By the law of cosines,

$s^2 = 4^2 + 8^2 - 2\cdot 8\cdot 4\cos\theta$

$\Rightarrow s^2 = 80 - 64\cos\theta$

We want to find $\frac{ds}{dt}$ when time = 1:00 P.M

(or) $\theta = \frac{2\pi}{12} = \frac{\pi}{6}$ rad

$2s\left(\frac{ds}{dt}\right) = +64\sin\theta\left(\frac{d\theta}{dt}\right)$

$\left(\frac{ds}{dt}\right) = \frac{32\sin\theta}{s}\left(\frac{d\theta}{dt}\right)$

when $\theta = \frac{\pi}{6}$, $\sin\theta = \frac{1}{2}$, $s = \sqrt{80 - 64\cos\frac{\pi}{6}} = \sqrt{80 - 32\sqrt{3}}$

$= 4\sqrt{5 - 2\sqrt{3}}$ mm

$$\Rightarrow \frac{ds}{dt} = \frac{32 \cdot 1/2}{4\sqrt{5-2\sqrt{3}}}\left(-\frac{11\pi}{6}\right)$$

$$= \frac{-176\pi}{24\sqrt{5-2\sqrt{3}}} = \frac{-22\pi}{3\sqrt{5-2\sqrt{3}}} \text{ mm/h}$$

$$\boxed{\frac{ds}{dt} = \frac{-22\pi}{3\sqrt{5-2\sqrt{3}}} \text{ mm/h}}$$

3.10

6) $f(x) = \dfrac{1}{\sqrt{2+x}}$, $a = 0$

$$L(x) = f(a) + f'(a)(x-a)$$

$$f'(x) = -\frac{1}{2}(x+2)^{-3/2}, \quad f'(a) = -\frac{1}{2}(2)^{-3/2} = -\frac{1}{4\sqrt{2}}$$

$$L(x) = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}}(x) = \frac{1}{\sqrt{2}}\left(1 - \frac{x}{4}\right)$$

$$\boxed{L(x) = \frac{1}{\sqrt{2}}\left(1 - \frac{x}{4}\right)}$$

8) $f(x) = (x)^{1/3}$, $a = -8$

$f'(x) = \frac{1}{3}(x)^{-2/3}$, $f'(a) = \frac{1}{3}(-8)^{-2/3} = \frac{1}{12}$

$$L(x) = f(a) + f'(a)(x-a)$$
$$= -2 + \frac{1}{12}(x+8) = \frac{x}{12} + \left(\frac{2}{3} - 2\right)$$
$$= \frac{x}{12} - \frac{4}{3}$$

$$\boxed{L(x) = \frac{x}{12} - \frac{4}{3}}$$

16) $y = \cos \pi x$

$\Rightarrow \boxed{dy = -\pi \sin \pi x \, dx}$

22) $y = x^3 - 6x^2 + 5x - 7$, $x = -2$, $dx = 0.1$

$\Rightarrow \underline{dy = (3x^2 - 12x + 5)dx}$

$x = -2$, $dx = 0.1 \Rightarrow dy = (12 + 24 + 5)(0.1)$
$$= (41)(0.1) = 4.1$$

$$\boxed{dy = 4.1}$$

24) $y = 1/(x+1)$, $x = 1$, $dx = -0.01$

$\Rightarrow dy = -1/(x+1)^2 \, dx$

$x = 1$, $dx = -0.01 \Rightarrow dy = -1/2^2(-0.01) = \frac{0.01}{4}$
$= 0.0025$

$\boxed{dy = 0.0025}$

32) $\sqrt{99.8} = \sqrt{100-0.2}$ $(x = 100, dx = -0.2)$

$y = \sqrt{x} = f(x)$

$\Rightarrow dy = \frac{1}{2\sqrt{x}} dx$

$x = 100, dx = -0.2 \Rightarrow dy = \frac{1}{2\sqrt{100}}(-0.2) = -\frac{0.2}{20}$
$= -0.01$

$\Rightarrow dy = -0.01$

$\sqrt{99.8} = f(99.8) = f(x+dx) \approx f(x) + dy = \sqrt{100} - 0.01$
$= 10 - 0.01$
$= 9.99$

$\Rightarrow \boxed{\sqrt{99.8} \approx 9.99}$

34) $1/1002 = 1/1000+2 \quad (x=1000, dx=2)$

$y = 1/x = f(x)$

$\Rightarrow dy = -1/x^2 dx$

$x=1000, dx=2 \Rightarrow dy = -1/10^6 \, 2 = -0.000002$

$1/1002 = f(x+dx) \approx f(x)+dy = 1/1000 - 0.000002$

$= 0.000998$

$$\boxed{1/1002 \approx 0.000998}$$

36) $\cos(31.5) = \cos(30+1.5) \quad (x=30, dx=1.5)$

$x = 30° = \frac{\pi}{6}$ rad, $dx = 1.5' = \frac{1.5\pi}{180}$ rad,

$y = \cos x = f(x)$

$dy = -\sin x \, dx$

$x=30, y=1.5 \Rightarrow dy = -\sin\frac{\pi}{6}(1.5)\pi/180$

$= -\frac{1}{2}\frac{\pi}{120} = -\frac{\pi}{240}$

$\cos(31.5) = f(x+dx) \approx f(x)+dy$

$= \cos\left(\frac{\pi}{6}\right) - \frac{\pi}{240}$

$= \frac{\sqrt{3}}{2} - \frac{\pi}{240}$

$$\Rightarrow \boxed{\cos(31.5) \approx \frac{\sqrt{3}}{2} - \frac{\pi}{240}}$$

38) $(1.01)^6 \approx 1.06$

$f(x) = x^6$

$f'(x) = 6x^5$

If $a=1$, $f'(a) = 6$

$L(x) = f(a) + f'(a)(x-a)$

$= 1 + 6(x-1) = 6x-5$

$L(x) = 6x-5$

$(1.01)^6 = f(1.01) \approx L(1.01) = 6(1.01) - 5$
$= 1.06$

$\Rightarrow \boxed{(1.01)^6 \approx 1.06}$