CALC I — MID.I — SOLUTIONS

Q1]...[13 points] Write down a limit definition of $f'(2)$ where $f(x) = x^3$. Also write down (in words) two interpretations of $f'(2)$.

$$f'(a) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right) \quad \text{OR} \quad \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right)$$

In our case $a = 2$ & so

$$f'(2) = \lim_{h \to 0} \left(\frac{f(2+h) - f(2)}{h} \right) \quad \text{OR} \quad \lim_{x \to 2} \left(\frac{f(x) - f(2)}{x - 2} \right)$$

Interp ① [GEOMETRY] $f'(2)$ = slope of tangent to $y = f(x)$ at $(2, f(2))$.

Interp ② [Analytical] $f'(2)$ = rate of change of $f(x)$ w.r.t. x at 2.

Compute the limit that you have written down above (that is compute $f'(2)$). Show all your work carefully.

$$f'(2) = \lim_{h \to 0} \left(\frac{f(2+h) - f(2)}{h} \right) = \lim_{h \to 0} \left(\frac{(2+h)^3 - 2^3}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\cancel{2^3} + 3(2^2)h + 3(2)h^2 + h^3 - \cancel{2^3}}{h} \right)$$

$$= \lim_{h \to 0} \left[\frac{\cancel{h}[3(2^2) + 3(2)h + h^2]}{\cancel{h}} \right] = \lim_{h \to 0} \left(3(4) + 6h + h^2 \right)$$

$$= 12 + 0 + 0 = \boxed{12}$$

Write down the equation of the tangent line to the graph $y = x^3$ at the point $(2, 8)$.

$f'(2) = 12$ = slope of tangent line

$(2, 8)$ = point on tangent line

if (x, y) is any other pt on tangent line we know

$$\frac{y - 8}{x - 2} = 12 \quad \text{OR} \quad \boxed{(y - 8) = 12(x - 2)}$$

Q2]... [**12 points**] Write down the values of the following two limits (you do **not** have to give proofs).

$$\lim_{x \to 0} \sin(x) = 0$$

$$\lim_{x \to 0} \cos(x) = 1$$

Write out the angle addition formula for the cosine function.

$$\cos(A + B) = \cos A . \cos B - \sin A . \sin B$$

Prove that the cosine function $\cos(x)$ is continuous at the input point a.

FORMULA ABOVE

$$\lim_{h \to 0} \cos(a+h) \stackrel{\downarrow}{=} \lim_{h \to 0} \left(\cos(a)\cos(h) - \sin(a)\sin(h) \right)$$

LIMIT LAWS

$$\stackrel{\downarrow}{=} \cos(a) \lim_{h \to 0} (\cos(h)) - \sin(a) \lim_{h \to 0} (\sin(h))$$

LIMITS above

$$\stackrel{\downarrow}{=} \cos(a) . 1 - \sin(a) . 0$$

$$= \cos(a)$$

We've shown ... $\lim_{h \to 0} (\cos(a+h)) = \cos(a)$

OR $\lim_{x \to a} (\cos(x)) = \cos(a)$

Thus $\cos(x)$ is continuous at a.

write $x = a+h \to a+0 = a$ as $h \to 0$

Q3]... [**12 points**] Find the values of the numbers a and b which make the functions below continuous.

$$f(x) = \begin{cases} 2x + a & \text{if } x \geq 1 \\ x^2 - 2x + 5 & \text{if } x < 1 \end{cases}$$

$$\lim_{x\to 1^+} \left(f(x)\right) = \lim_{x\to 1^+} (2x+a) = 2(1) + a = \boxed{2+a} \overset{f(1)}{\parallel}$$

$$\lim_{x\to 1^-} \left(f(x)\right) = \lim_{x\to 1^-} (x^2 - 2x + 5) = 1^2 - 2(1) + 5 = \boxed{4}$$

For continuity we want $\lim_{x\to 1} (f(x))$ to <u>exist</u> (& to equal $f(1)$)

$\Rightarrow$ want both boxes above to agree!

$\Rightarrow 4 = 2 + a \quad \Rightarrow \boxed{2 = a}$

$$g(x) = \begin{cases} \frac{x^3-8}{x-2} & \text{if } x \neq 2 \\ b & \text{if } x = 2 \end{cases}$$

$$\lim_{x\to 2} \left(g(x)\right) = \lim_{x\to 2} \left(\frac{x^3-8}{x-2}\right) = \lim_{x\to 2} \left(\frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}}\right)$$

$$= \lim_{x\to 2} (x^2 + 2x + 4)$$

$$= 2^2 + 2(2) + 4 = \boxed{12}$$

$g(2) = \boxed{b}$

These must agree for continuity of $g(x)$

$\Rightarrow \boxed{b = 12}$

Q4]... [**13 points**] Compute the following limits.

$$\lim_{x\to 1} \frac{\frac{1}{x^2} - 1}{x - 1}$$

$$= \lim_{x\to 1} \left(\frac{\frac{1}{x^2} - \frac{x^2}{x^2}}{x-1} \right) = \lim_{x\to 1} \left(\frac{\frac{1-x^2}{x^2}}{x-1} \right)$$

$$= \lim_{x\to 1} \left(\frac{(1-x^2)}{x^2(x-1)} \right) = \lim_{x\to 1} \left(\frac{\cancel{(1-x)}^{-1}(1+x)}{x^2 \cancel{(x-1)}} \right)$$

$$= \lim_{x\to 1} \left(\frac{-(1+x)}{x^2} \right) = -\frac{2}{1^2} = \boxed{-2}$$

$$\lim_{x\to 8} \frac{x - 8}{\sqrt[3]{x} - 2}$$

$$= \lim_{x\to 8} \left(\frac{(\sqrt[3]{x})^3 - 2^3}{\sqrt[3]{x} - 2} \right) = \lim_{x\to 8} \left(\frac{\cancel{(\sqrt[3]{x} - 2)}\left((\sqrt[3]{x})^2 + (2)(\sqrt[3]{x}) + (2)^2\right)}{\cancel{(\sqrt[3]{x} - 2)}} \right)$$

$$= \lim_{x\to 8} \left((\sqrt[3]{x})^2 + (2)(\sqrt[3]{x}) + (2)^2 \right)$$

$$= (2)^2 + (2)(2) + (2)^2 = \boxed{12}$$