SOLUTIONS – M II

Q1]...[13 points] Compute the derivatives of the following functions. Write down the names of key differentiation rules you are using. You do **not** have to spend time simplifying the expressions in your answers.

$$\sin^3(5x+4)$$

Chain Rule : let $u = 5x+4$, $v = \sin(u)$, $y = v^3$

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx} = (3v^2)(\cos(u))(5)$$

$$= 3\sin^2(5x+4) . \cos(5x+4) . 5$$

$$3x^\pi - \pi x^3$$

Power Rule

$$y' = (3)(\pi)x^{\pi-1} - (\pi)(3)x^{3-1}$$

$$\frac{\sin(x)}{x^4+7x}$$

Quotient Rule

$$y' = \frac{\frac{d(\sin x)}{dx}(x^4+7x) - \sin x \frac{d}{dx}(x^4+7x)}{(x^4+7x)^2}$$

$$= \frac{\cos(x)(x^4+7x) - \sin(x)(4x^3+7)}{(x^4+7x)^2}$$

Q2]...[12 points] Give a proof of the product rule $(fg)' = f'g + fg'$.

$$(fg)'(x) = \lim_{h\to 0}\left(\frac{f(x+h)g(x+h) - f(x)g(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}\right)$$

$$= \lim_{h\to 0}\left[\left(\frac{f(x+h)-f(x)}{h}\right)g(x+h) + f(x)\left(\frac{g(x+h)-g(x)}{h}\right)\right]$$

$$= f'(x)g(x) + f(x)g'(x)$$

Compute the derivative y' of the following function. You do *not* have to simplify your answer.

$$y = (2x^3 - 4)(7\tan(x) + 1)$$

$$y' = \frac{d}{dx}(2x^3-4)\,(7\tan(x)+1) + (2x^3-4)\frac{d}{dx}(7\tan(x)+1)$$

$$= (6x^2)(7\tan(x)+1) + (2x^3-4)(7\sec^2(x))$$

Q3]...[12 points] Write down the following limits.

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} = 0$$

Compute the derivative $f'(x)$ of the function $f(x) = \sin(x)$ using the limit definition of the derivative.

$$f'(x) = \lim_{h \to 0} \left(\frac{\sin(x+h) - \sin(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \right)$$

$$= \lim_{h \to 0} \left[\sin(x) \left(\frac{\cos(h) - 1}{h} \right) + \cos(x) \left(\frac{\sin(h)}{h} \right) \right]$$

$$= \sin(x) \cdot 0 + \cos(x) \cdot 1 = \cos(x)$$

Suppose that the position of an object on the x-axis at time t is given by $x(t) = \sin(5t + \pi)$.
Compute the object's velocity and acceleration at time t.
What is the maximum value of the objects acceleration?

$$\text{Velocity} = \frac{dx}{dt} = \frac{d}{dt} \sin(5t+\pi) = 5\cos(5t+\pi)$$

$$\text{acceleration} = \frac{d(\text{vel})}{dt} = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(5\cos(5t+\pi)\right) = -25\sin(5t+\pi)$$

Max value of acc. occurs at min value of sin function which is -1

$$\Rightarrow \text{Max acc.} = (-25)(-1) = \boxed{25}$$

Q4]... **[13 points]** Show that the curves $xy = 10$ and $x^2 - y^2 = 10$ meet (intersect) at right angles. That is, show that their slopes are perpendicular at the points of intersection.

$$\frac{d}{dx}(xy) = \frac{d}{dx}(10)$$

$$\frac{dx}{dx}y + x\frac{dy}{dx} = 0$$

$$1y + x\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{y}{x}}$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = \frac{d(10)}{dx}$$

$$2x - 2y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-2x}{-2y} = \frac{x}{y}}$$

Product of slopes at an intersection point (x,y) is given by $\left(\frac{-y}{x}\right)\left(\frac{x}{y}\right) = -1$

$\Rightarrow$ perpendicular !

Compute the n-th derivative $f^{(n)}(x)$ of the function

$$f(x) = \frac{1}{(1+5x)}$$

$$f(x) = (1+5x)^{-1}$$

$$f'(x) = (-1)(1+5x)^{-2}(5)$$

$$f''(x) = (-1)(-2)(1+5x)^{-3}(5)(5)$$

$$f'''(x) = (-1)(-2)(-3)(1+5x)^{-4}(5)(5)(5)$$

Pattern $\vdots$

$$f^{(n)}(x) = (-1)(-2)\cdots(-n)(1+5x)^{-(n+1)}(5)\cdots(5)$$

$$= (-1)^n n!(5)^n (1+5x)^{-(n+1)}$$

$$\boxed{f^{(n)}(x) = \frac{(-1)^n 5^n n!}{(1+5x)^{n+1}}}$$