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Calculus III [2433–001] Final Examination

Wednesday, June 9, 1999

For full credit, give reasons for all your answers.

Q1]...[15 points] Write down the equation of the line through the point $(2, -1, 1)$ which is parallel to the vector $\langle 1, 2, 3\rangle$.
Write down the equation of the plane through the point $(1, 1, 1)$ which is perpendicular to the line above.
Write down the equation of any plane which is perpendicular to the plane $2x - 3y + 4z = 17$ and verify that the two planes are indeed perpendicular.
Q2]...[22 points] Sketch the polar curves $r = \sin\theta$ and $r = 1 - \sin\theta$ on the same graph, and compute (and draw in) their points of intersection.
Compute the area which is common to both curves $r = \sin\theta$ and $r = 1 - \sin\theta$ above.
Find the arclength of the portion of the curve $r = \sin\theta$ which lies outside of the curve $r = 1 - \sin\theta$.
Q3]...[20 points] Compute the McLaurin series for the function $f(x) = \ln(3 + x)$. Write dow the general term in your series.
What are the radius and interval of convergence of the series above?
Write down the power series for the function $g(x) = \ln(3 - x^2)$. What is it's radius of convergence?
Q4]...[21 points] Use the various series tests learned in class to determine whether each of the following are *absolutely convergent*, *conditionaly convergent*, or *divergent*.

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{2n-1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}2^n}$$

Q5]...[20 points] At time $t = 0$ a ball is kicked horizontallyh off a cliff of height 200ft with an initial speed of 40ft/sec. Assume that the only force acting on the ball is due to gravity, and that produces an acceletation of 32ft/sec^2 vertically downwards. See the diagram.
Compute $\mathbf{r}(t)$, the position vector of the ball at time t.
Find the time taken for the ball to reach the ground.
Compute the horizontal distance that the ball has travelled during this time.
Write down an expression for the **total distance** the ball travels through the air (you do not have to evaluate this expression).
Q6]...[22 points] Compute the curvature $k(x)$ of the graph of $y = \sin x$ at the point $(x, \sin x)$.
Find the points where $k(x)$ has local maxima/minima. Indicate these points on a graph of $y = \sin x$.
Suppose that a point with position vector $\mathbf{r}(t)$ moves around on a sphere of radius 3 centered on the origin in $\mathbf{R}^3$. The point does not necessarily move in a circle. Show that it's velocity $\mathbf{v}(t)$ is always perpendicular to $\mathbf{r}(t)$.

— 1999 — Solns

Q1 $\vec{r} = \langle 2,-1,1\rangle + t\langle 1,2,3\rangle$

$x = 2+t$
$y = -1+2t$
$z = 1+3t$

$\langle 1,2,3\rangle \cdot \langle x-1, y-1, z-1\rangle = 0$

$1(x-1) + 2(y-1) + 3(z-1) = 0$

Our plane has normal $\langle 2,-3,4\rangle$

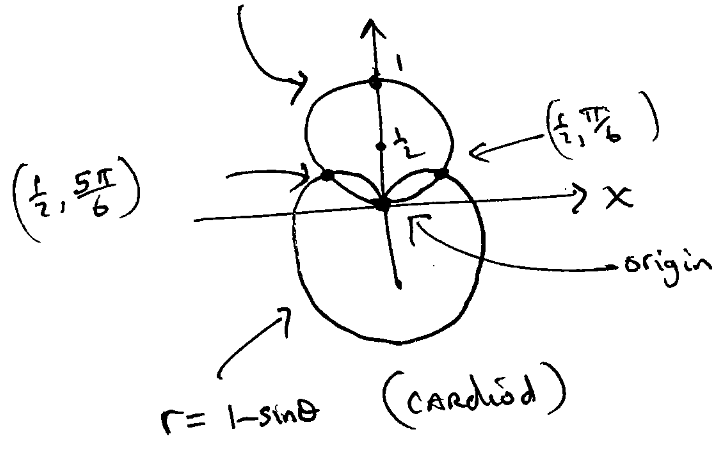
$\langle 3,2,0\rangle$ is $\perp$ to this

let plane go through (0,0,0) ---- so $3(x-0) + 2(y-0) + 0(z-0) = 0$

$$\boxed{3x + 2y = 0}$$

Q2 $\boxed{r = \sin\theta} \Rightarrow r^2 = r\sin\theta \Rightarrow x^2+y^2 = y \Rightarrow x^2 + (y-\tfrac{1}{2})^2 = \tfrac{1}{2}^2$

circle center $(0,\tfrac{1}{2})$ rad $\tfrac{1}{2}$



intersect at

$\sin\theta = 1-\sin\theta$

$2\sin\theta = 1$

$\sin\theta = \tfrac{1}{2}$

$$\boxed{\theta = \pi/6, 5\pi/6 \qquad r = \tfrac{1}{2}}$$

& intersect at origin

Common Area (overlap area) is 

$= 2\Big($  $\Big)$

$$\text{Area} = 2\left(\int_0^{\pi/6} \frac{r^2}{2}\, d\theta + \int_{\pi/6}^{\pi/2} \frac{r^2}{2}\, d\theta \right)$$

$\uparrow$ use $r = \sin\theta$ $\qquad\qquad$ $\uparrow$ use $r = 1 - \sin\theta$

$$= \int_0^{\pi/6} \sin^2\theta\, d\theta + \int_{\pi/6}^{\pi/2} \left(\sin^2\theta + 1 - 2\sin\theta\right) d\theta$$

$$= \int_0^{\pi/6} \frac{1-\cos(2\theta)}{2}\, d\theta + \int_{\pi/6}^{\pi/2} \left(\frac{1-\cos 2\theta}{2} + 1 - 2\sin\theta\right) d\theta$$

$$= \int_0^{\pi/2} \frac{1-\cos(2\theta)}{2}\, d\theta + \int_{\pi/6}^{\pi/2} \left(1 - 2\sin\theta\right) d\theta$$

$$= \left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_0^{\pi/2} + \Big[\theta + 2\cos\theta\Big]_{\pi/6}^{\pi/2}$$

$$= \pi/4 + (\pi/2 + 0) - \left(\pi/6 + 2\left(\frac{\sqrt{3}}{2}\right)\right) = \boxed{\frac{7\pi}{12} - \sqrt{3}}$$

$$L = 2\int_{\pi/6}^{\pi/2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= 2\int_{\pi/6}^{\pi/2}\sqrt{\sin^2\theta + \cos^2\theta}\,d\theta$$

$$= 2\int_{\pi/6}^{\pi/2} d\theta = 2\theta\Big]_{\pi/6}^{\pi/2} = 2\left(\pi/2 - \pi/6\right)$$

$$= 2\left(\frac{2\pi}{6}\right) = \boxed{\frac{2\pi}{3}}$$

Q3

$f(x) = \ln(3+x)$	$f(0) = \ln(3)$	
$f'(x) = \frac{1}{3+x}$	$f'(0) = \frac{1}{3}$	$\frac{f'(0)}{1!} = \frac{1}{3}$
$f''(x) = \frac{-1}{(3+x)^2}$	$f^{(2)}(0) = \frac{-1}{3^2}$	$\frac{f^{(2)}(0)}{2!} = \frac{-1}{2.3^2}$
$f'''(x) = \frac{(-1)(-2)}{(3+x)^3}$	$f^{(3)}(0) = \frac{2!}{3^3}$	$\vdots$
	$\vdots$	
$f^{(n)}(x) = \frac{(-1)(-2)\cdots(-(n-1))}{(3+x)^n}$	$f^{(n)}(0) = \frac{(-1)^{n-1}(n-1)!}{3^n}$	$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n-1}}{n.3^n}$

$$\ln(3+x) = f(x) = \ln(3) + \frac{x}{3} - \frac{x^2}{2.3^2} + \frac{x^3}{3.3^3} - \cdots + \frac{(-1)^{n-1}x^n}{n.3^n} + \cdots$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\overset{|x|}{\cancel{|x|^{n+1}}}}{(n+1)\underset{3}{\cancel{3^{n+1}}}} \cdot \frac{n.\cancel{3^n}}{\cancel{|x|^n}}$$

$$= \frac{|x|}{3} \quad \frac{n}{n+1} \longrightarrow \frac{|x|}{3} \quad \text{as } n \to \infty$$

Ratio Test $\Rightarrow$

cgt for $\frac{|x|}{3} < 1$ $\quad |x| < 3$ $\quad -3 < x < 3$

& divgt for $\frac{|x|}{3} > 1$ $\quad |x| > 3$.

at $x=3$ series becomes

$$\ln(3) + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - - - - -$$

convgt (A.S.T. (alternating harmonic))

at $x=-3$ series becomes

$$\ln(3) \underbrace{-\frac{1}{3} - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - - - - -}$$

divgt $\ominus$ harmonic.

$\Rightarrow$ Radius of convergence is 3

& interval of convergences is $(-3, 3]$.

$x \longleftrightarrow -x^2$

$$\ln\left(3-x^2\right) = \ln(3) - \frac{x^2}{3} \mp \frac{x^4}{2.3^2} - \frac{x^6}{3.3^3} - \frac{x^8}{4.3^4} - - -$$

$$= \ln(3) - \sum_{n=1}^{\infty} \frac{x^{2n}}{n.3^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{2n+2}}{(n+1)\,3^{n+1}} \cdot \frac{n.3^n}{|x|^{2n}}$$

$$= \frac{|x|^2}{3} \cdot \frac{n}{n+1} \rightarrow \frac{|x|^2}{3}$$

Ratio Test $\Rightarrow \left[\begin{array}{l} \underline{\text{cgt}}\ |x|^2 < 3, \quad |x| < \sqrt{3} \\ \underline{\text{dvgt}}\ |x|^2 > 3, \quad x > \sqrt{3} \end{array}\right.$

Rad of Convergence is $\sqrt{3}$

Q4 $\frac{|3^{n+1}|}{|(n+1)!|} \frac{|n!|}{|3^n|} = \frac{3}{n+1} \to 0 < 1$

Abs convgt by Ratio Test

$\sum \frac{(-1)^n}{2n-1}$ cgt A.S.T. $\frac{1}{2x+1} \downarrow 0$

$\left|\frac{(-1)^n}{2n-1}\right| = \sum \frac{1}{2n-1}$ dvgt (Limit comp test with $\sum \frac{1}{n}$ harmonic series dvgt)

$\frac{1}{2n-1} \cdot \frac{n}{1} = \frac{1}{2 - \frac{1}{n}} \to \frac{1}{2} \begin{cases} \neq 0 \\ \neq \infty \end{cases}$

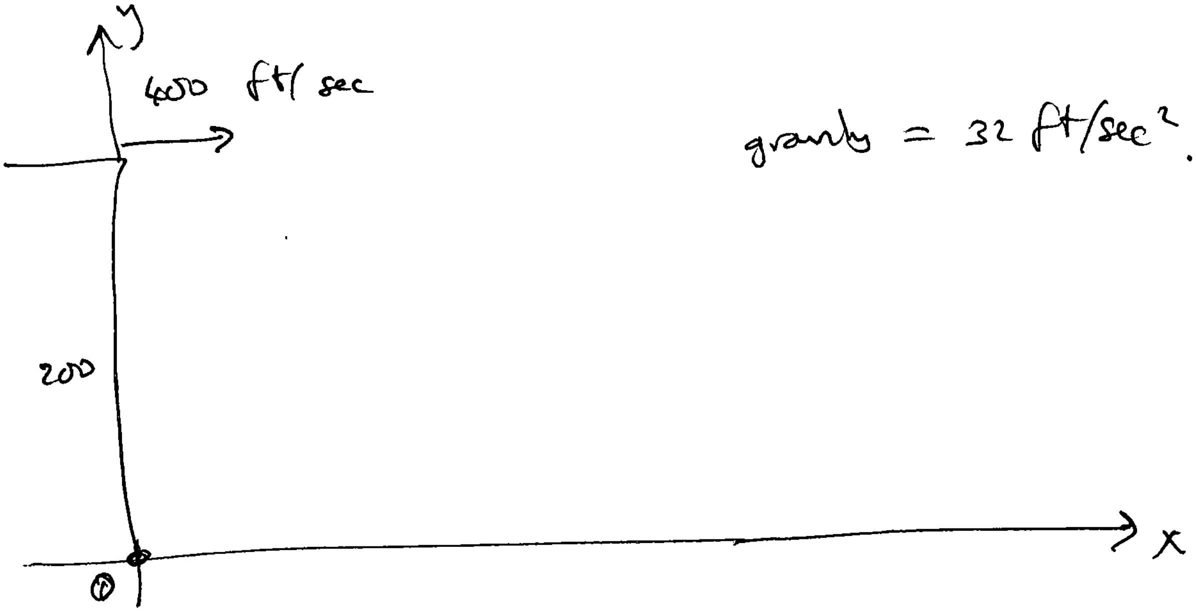
$\therefore$ Series is <u>cond</u> convgt.

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{\sqrt{n+1}\, 2^{n+1}} \cdot \frac{\sqrt{n}\, 2^n}{1} = \frac{1}{2} \frac{\sqrt{n}}{\sqrt{n+1}}$

$\longrightarrow \frac{1}{2} < 1 \Rightarrow$ cgt.

Abs Cgt by Ratio Test

Q5



gravity = 32 ft/sec^2.

Know $\vec{r}''(t) = \langle 0, -32 \rangle = 0\hat{i} - 32\hat{j}$

$\int dt \Rightarrow \vec{r}'(t) = \vec{C} + \langle 0, -32t \rangle$

at $t=0$, $\vec{r}'(0) = 400\hat{i} = \langle 400, 0 \rangle$

$\Rightarrow \vec{C} = \langle 400, 0 \rangle$

$\vec{r}'(t) = \langle 400, 0 \rangle + \langle 0, -32t \rangle$

$= \langle 400, -32t \rangle$

$\int dt \Rightarrow \vec{r}(t) = \vec{D} + \langle 400t, -16t^2 \rangle$

At $t=0$, $\vec{r}(0) = \langle 0, 200 \rangle \Rightarrow \vec{D} = \langle 0, 200 \rangle$

$$\boxed{\vec{r}(t) = \langle 400t,\ 200 - 16t^2 \rangle}$$

Ball reaches ground when y-coord = 0 (ground level)

$$200 - 16t^2 = 0$$

$$t^2 = \frac{200}{16}$$

$$t = \sqrt{\frac{200}{16}} = \frac{\sqrt{2}\ 10}{4} = \frac{5\sqrt{2}}{2}$$

$\frac{5\sqrt{2}}{2}$ seconds

Horizontal distance traveled $= 400\,\frac{5\sqrt{2}}{2} = 200\ 5\sqrt{2}$

$= 1000\sqrt{2}$ ft.

Total distance = Arc length of path r(t)

$$= \int_0^{\frac{5\sqrt{2}}{2}} \sqrt{\left(\frac{d(400t)}{dt}\right)^2 + \left(\frac{d}{dt}(200-16t^2)\right)^2}\, dt$$

$$\boxed{\text{Total Dist.} = \int_0^{\frac{5\sqrt{2}}{2}} \sqrt{(400)^2 + (32t)^2}\, dt}$$

Q6

$$K = \frac{|r' \times r''|}{|r'|^3}$$

$y = f(x)$

$\Rightarrow r = \langle x, f(x), 0 \rangle$

$r' = \langle 1, f'(x), 0 \rangle$

$r'' = \langle 0, f''(x), 0 \rangle$

$r' \times r'' = \langle 0, 0, f''(x) \rangle$

$$\leftarrow \quad = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}$$

$y = \sin x$

$y' = \cos x$

$y'' = -\sin x$

$$K(x) = \frac{|-\sin x|}{\left[1 + \cos^2(x)\right]^{3/2}}$$

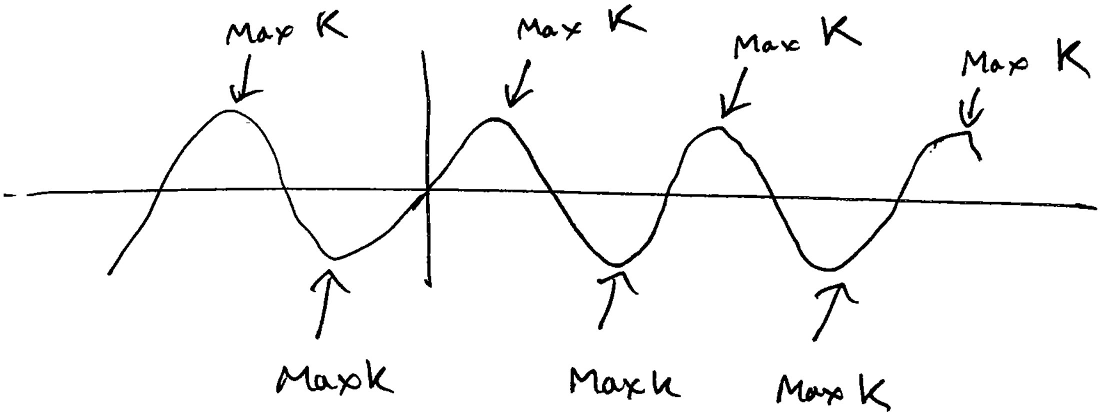
$$K(x) = \frac{|\sin(x)|}{\left[1 + \cos^2(x)\right]^{3/2}}$$

Max numerator $= 1$ occurs at $x = \pi/2, 3\pi/2, ---$

$=$ Min denom. $= 1$.

$K(x) = 1$ Max

Min numerator $= 0$ occurs when ~~denom~~ $x = 0, \pi, 2\pi, ----$



Min K points are all x-intercepts.

$|\vec{r}(t)| = 3$

$\vec{r}(t) \cdot \vec{r}(t) = 9$

$\frac{d}{dt} \Rightarrow \quad \frac{d}{dt}(\vec{r} \cdot \vec{r}) = \frac{d}{dt}(9) = 0$

$\Rightarrow 2\,\vec{r} \cdot \frac{d\vec{r}}{dt} = 0$

$\Rightarrow \vec{r} \cdot \frac{d\vec{r}}{dt} = 0$

$\Rightarrow \vec{r}(t) \perp \frac{d\vec{r}(t)}{dt}$
