MID I
Solutions

Q1]...[12 points] Find a **disjunctive normal form** expression (involving $\wedge$, $\vee$, $\neg$, and P, Q, R) which has the following truth table. Show the steps of your work.

P	Q	R		
T	T	T	T	$\leftarrow P \wedge Q \wedge R$
T	T	F	F	
T	F	T	T	$\leftarrow P \wedge \neg Q \wedge R$
T	F	F	F	
F	T	T	F	
F	T	F	F	
F	F	T	T	$\leftarrow \neg P \wedge \neg Q \wedge R$
F	F	F	T	$\leftarrow \neg P \wedge \neg Q \wedge \neg R$

Step①: Find expressions whose truth tables give a T in the appropriate row & F's elsewhere. There are 4 expressions for this example

Step②: Take the disjunction of the expressions in step①.

$$dnf = (P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$$

Find a **conjunctive normal form** expression (involving $\wedge$, $\vee$, $\neg$, and P, Q, R) which has the same truth table above. Show the steps of your work.

Step①: Negate the output column $\longrightarrow$ F T F T T T F F

Step②: Write dnf for this new output col:

$$(P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R)$$

Step③ Negate this dnf ... use deMorgan ... get cnf for original table!

$$(\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (P \vee \neg Q \vee R)$$

Q2]...[11 points] Write down the distributive law for $\wedge$ over $\vee$.

$$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$$

Write down the distributive law for $\vee$ over $\wedge$.

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

Write down the two De Morgan laws (involving negations of $\wedge$ and $\vee$ statements).

$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$

$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$

Use the De Morgan and distributive laws to show that the expression

$$[P \wedge (\neg Q) \wedge R] \vee [P \wedge (\neg Q) \wedge (\neg R)] \vee [P \wedge Q \wedge R] \vee \neg[(\neg P) \vee (\neg Q) \vee R]$$

is logically equivalent to P.

de Morgan

$$\text{Expression} \equiv (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R)$$

factor out a "P" (distrib laws)

$$\equiv P \wedge \left[\underbrace{(\neg Q \wedge R) \vee (\neg Q \wedge \neg R)}_{\downarrow \text{ factor } \neg Q} \vee \underbrace{(Q \wedge R) \vee (Q \wedge \neg R)}_{\downarrow \text{ factor } Q} \right]$$

$$\equiv P \wedge \left[(\neg Q \wedge (R \vee \neg R)) \vee (Q \wedge (R \vee \neg R)) \right]$$

$$\equiv P \wedge \left[(\neg Q \wedge \top) \vee (Q \wedge \top) \right]$$

$$\equiv P \wedge (\neg Q \vee Q)$$

$$\equiv P \wedge \top$$

$$\equiv P \qquad \text{done!}$$

Q3]... [12 points] Are the following two expressions logically equivalent. If you say so, please explain why. If you say not, then please give an example which shows that they are different.

$$\forall x[P(x) \to Q(x)]$$

and

$$(\forall x P(x)) \to (\forall x Q(x))$$

No Example (in class!) Universe = all integers

$P(x)$ = x is even

$Q(x)$ = x is odd

Statement ① is false: eg 2 is even, but not odd.

Statement ② is automatically true, since the hypothesis "every integer is even" is false.

Same question for the expressions

$$\exists x[P(x) \vee Q(x)]$$

and

$$(\exists x P(x)) \vee (\exists x Q(x))$$

Yes $\boxed{1^{st} \to 2^{nd}}$

$\exists x$ such that $P(x) \vee Q(x)$ true

implies $P(x)$ true or $Q(x)$ true for this value of x

$\Rightarrow \exists x\ P(x)$ or $\exists x\ Q(x)$. $\Rightarrow \exists x P(x) \vee \exists x Q(x)$.

$\boxed{2^{nd} \to 1^{st}}$

$\exists x\ P(x) \vee \exists x\ Q(x)$

$\Rightarrow \exists x\ P(x)$ | or $\exists x\ Q(x)$

$\Rightarrow P(x)$ true | $\Rightarrow Q(x)$ true ...... (for a possibly different x than in left column)

$\Rightarrow P(x) \vee Q(x)$ true | $\Rightarrow P(x) \vee Q(x)$ true

$\hookrightarrow \exists x\ (P(x) \vee Q(x))$.

Q4]...[15 points] Give a **direct proof** of the following. *If m and n are odd integers, then their product is also odd.*

m odd $\Rightarrow$ $m = 2k+1$ for some integer k.

n odd $\Rightarrow$ $n = 2l+1$ for some integer l.

$$\Rightarrow mn = (2k+1)(2l+1) = 4kl + 2k + 2l + 1$$
$$= 2(2kl + k + l) + 1$$

which is of the form $2(\text{integer}) + 1$

$\Rightarrow$ is odd.

$\boxtimes$

Write down the contrapositive of the following statement about integers n. *If n^3 is even, then n is also even.*

If n is odd, then n^3 is odd.

Prove the statement *"If n^3 is even, then n is also even"* by giving a proof of its contrapositive.

Start with n is odd

$\Rightarrow$ $n^2 = n.n$ = product of 2 odd integers is odd (by 1st part above)

$\Rightarrow$ $n^3 = n^2.n$ = product of 2 odd integers is odd (by 1st part above)

$\Rightarrow$ n^3 odd.

$\boxtimes$

Q5]...[15 points] Give a proof of the following: *The cube root of 2 is irrational.* You are free to cite the results of **Q4** if they are of any help to you.

Proof by contradiction.

Assume $\sqrt[3]{2}$ is rational.

Thus $\sqrt[3]{2} = p/q$ for $p, q \in \mathbb{Z}^+$.

By dividing numerator + denom by some power of 2, we may assume that at least one of p, q is odd. (*)

$$2 = \frac{p^3}{q^3}$$

$$2q^3 = p^3$$

LHS is even $\Rightarrow p^3$ is even

$\Rightarrow p$ is even -- by Q4.

Writing $p = 2k$ for some integer k, we get

$$2q^3 = (2k)^3 = 8k^3$$

$$\Rightarrow q^3 = 4k^3$$

RHS is even $\Rightarrow q^3$ is even

$\Rightarrow q$ is even -- by Q4.

So we have both p and q are even $\Rightarrow$ contradicts (*).

So $\sqrt[3]{2}$ must be irrational. $\boxed{\text{FG}}$

Q6]...[20 points] State the principle of induction. $P(n)$ = statement involving ⊕ integer n.

$$\left.\begin{array}{l}\bullet\ P(1) \text{ true} \\ \bullet\ \forall k \left(P(k) \Longrightarrow P(k+1)\right)\end{array}\right] \Longrightarrow \quad P(n) \text{ true} \quad \forall n \in \mathbb{Z}^+.$$

Give a proof by induction of the following. *For each positive integer n,*

$$P(n): \qquad 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof

① $\underline{P(1) \text{ is true}}$: $\quad 1^2 = 1 \overset{??}{=} \frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = 1$

$1 = 1$ ✓ true.

② $\underline{\forall k \left[P(k) \Rightarrow P(k+1)\right]}$: Assume $P(k)$ true :

$$1^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Then $1^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{(k+1)\left(k(2k+1) + 6(k+1)\right)}{6}$$

$$= \frac{(k+1)\left(2k^2 + 7k + 6\right)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)\big((k+1)+1\big)\big(2(k+1)+1\big)}{6}$$

& so $P(k+1)$ holds.

By the principle of induction, $P(n)$ true $\forall n \in \mathbb{Z}^+$

$\boxed{//}$

Q7]...[15 points] Give a proof by induction of the following. $2^{2n-1} + 3^{2n-1}$ *is a multiple of* 5 *for all integers* $n \geq 1$.

$P(n)$: $2^{2n-1} + 3^{2n-1}$ is a multiple of 5

P(1) true : $2^{2(1)-1} + 3^{2(1)-1} = 2^1 + 3^1 = 2+3 = 5$ is a multiple of 5 ... (5(1)).

$\forall k \, (P(k) \Rightarrow P(k+1))$: Assume $P(k)$ true.

$2^{2k-1} + 3^{2k-1} = 5M$ for some integer M.

Now $2^{2(k+1)-1} + 3^{2(k+1)-1} = 2^{2k-1+2} + 3^{2k-1+2}$

$= 2^2 . 2^{2k-1} + 3^2 . 3^{2k-1}$

$= 4 . 2^{2k-1} + \underbrace{9}_{=(4+5)} . 3^{2k-1}$

$= 4\left(2^{2k-1} + 3^{2k-1}\right) + 5 . 3^{2k-1}$

$= 4(5M) + 5 \, 3^{2k-1}$ ... by P(k) true.

$= 5(4M + 3^{2k-1})$

$=$ multiple of 5. $\Rightarrow P(k+1)$ true.

By induction, P(n) true $\forall n \in \mathbb{Z}^+$.