

2 USEFUL LEMMAS

(1)

Lemma ① Let $H: X \times I \longrightarrow X$ be a deformation retraction of X onto $A \subseteq X$.

Let $h: X \rightarrow Y$ be a homeomorphism which takes A to $h(A) = B \subseteq Y$.

Then $h \circ H \circ (h \times 1)^{-1}: Y \times I \longrightarrow Y$

is a deformation retraction of Y onto $B \subseteq Y$.

Pf. sketch \searrow
It's clear that $h \circ H \circ (h \times 1)^{-1} = K$ is cts.

Now... Check [

- $K(y, 0) = y \quad \forall y \in Y$
- $K(y, 1) \in B \quad \forall y \in Y$
- $K(b, t) = b \quad \forall t \in I, \forall b \in B$.

Lemma ① is used multiple times in the assignment on (p,q) -knots.

Lemma ② tells us when a deformation retraction $H: X \times I \rightarrow X$ induces a deformation retraction of quotient spaces $K: Y \times I \rightarrow Y$ (where $q: X \rightarrow Y$ is quotient).

The equivalence relation needs to interact nicely with H .

Def ① Let $H: X \times I \rightarrow X$ be a deformation retraction. Say that the equivalence relation \sim on X is H-closed if

$$(*) \quad x \sim x' \implies H(x, t) \sim H(x', t) \quad \forall t \in I$$

Lemma ② Let $H: X \times I \rightarrow X$ be a deformation retraction of X onto $A \subseteq X$.

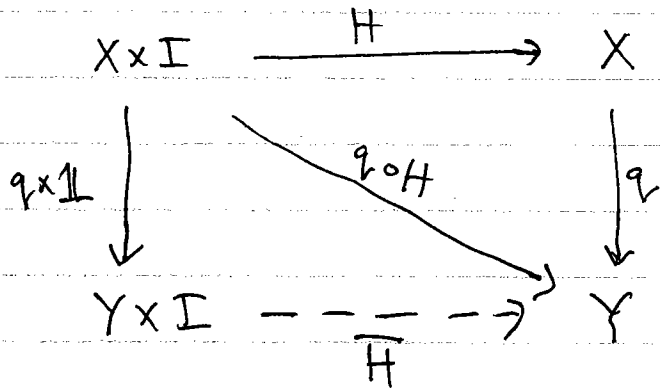
Let \sim be an H-closed equivalence relation on X .

Let $q: X \rightarrow X/\sim \equiv Y$ be the quotient space

& let $B = q(A)$.

Then $\exists \bar{H}: Y \times I \rightarrow Y$, a deformation retraction of Y onto B .

Pf sketch



$q: X \rightarrow Y$ quotient $\Rightarrow q \times \mathbb{1}: X \times I \rightarrow Y \times I$ is quotient
 (since I is locally compact, Hausdorff).

Now $q \circ H: X \times I \rightarrow Y$ is cts & ...

$$(q \times \mathbb{1})(x, t) = (q \times \mathbb{1})(x', t')$$

$$\Rightarrow (q(x), t) = (q(x'), t')$$

$$\Rightarrow q(x) = q(x') \text{ \& } t = t'$$

$$\Rightarrow t = t' \text{ and } x \sim x'$$

$$\Rightarrow H(x, t) \sim H(x', t) \text{ --- by property (*) of } \sim$$

$$\quad \quad \quad \parallel$$

$$\quad \quad \quad H(x', t')$$

$$\Rightarrow q(H(x, t)) = q(H(x', t'))$$

$\Rightarrow q \circ H$ is constant on the fibers of $(q \times \mathbb{1})$.

$\Rightarrow \exists$ well-defined, cts map $\bar{H}: Y \times I \rightarrow Y$

$$\bar{H}(q(x), t) = q(H(x, t))$$

Now check that

• $\bar{H}(y, 0) = y$, $\forall y \in Y$

• $\bar{H}(y, 1) \in B$, $\forall y \in Y$

• $\bar{H}(b, t) = b$, $\forall t \in I, \forall b \in B$.
