2 USEFUL LEMMAS

Lemma 1 Let $H : X \times I \longrightarrow X$ be a deformation retraction of X onto $A \subseteq X$.

Let $h : X \to Y$ be a homeomorphism which takes A to $h(A) = B \subseteq Y$.

Then $h \circ H \circ (h \times \mathbb{1})^{-1} : Y \times I \longrightarrow Y$

is a deformation retraction of Y onto $B \subseteq Y$.

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pf. sketch

It's clear that $h \circ H \circ (h \times \mathbb{1})^{-1} = K$ is cts.

Now... Check

- $K(y, 0) = y \quad \forall y \in Y$
- $K(y, 1) \in B \quad \forall y \in Y$
- $K(b, t) = b \quad \forall t \in I, \ \forall b \in B$.

Lemma 1 is used multiple times in the assignment on (p,q)-knots.

Lemma 2 tells us when a deformation retraction
$H: X \times I \to X$ induces a deformation retraction
of quotient spaces $K: Y \times I \longrightarrow Y$ (where $q: X \to Y$ is quotient).

The equivalence relation needs to interact nicely with H...

Def 1 Let $H: X \times I \longrightarrow X$ be a deformation retraction. Say that the equivalence relation $\sim$ on X is <u>H-closed</u> if

$$(*) \quad \boxed{x \sim x' \implies H(x,t) \sim H(x',t) \quad \forall t \in I}$$

Lemma 2 Let $H: X \times I \to X$ be a deformation retraction of X onto $A \subseteq X$.

Let $\sim$ be an H-closed equivalence relation on X.

Let $q: X \to X/\sim \equiv Y$ be the quotient space

& let $B = q(A)$.

Then $\exists\, \bar{H}: Y \times I \longrightarrow Y$, a deformation retraction of Y onto B.

<u>Pf sketch</u>

$$\begin{tikzcd}
X \times I \arrow[r, "H"] \arrow[d, "q \times \mathbb{1}"'] \arrow[dr, "q \circ H"] & X \arrow[d, "q"] \\
Y \times I \arrow[r, dashed, "\overline{H}"'] & Y
\end{tikzcd}$$

$q : X \to Y$ quotient $\Rightarrow$ $q \times \mathbb{1} : X \times I \to Y \times I$ is quotient (since I is locally compact, Hausdorff).

Now $q \circ H : X \times I \to Y$ is cts & ...

$$(q \times \mathbb{1})(x, t) = (q \times \mathbb{1})(x', t')$$

$$\Rightarrow (q(x), t) = (q(x'), t')$$

$$\Rightarrow q(x) = q(x') \;\&\; t = t'$$

$$\Rightarrow t = t' \text{ and } x \sim x'$$

$$\Rightarrow H(x, t) \sim \underset{\displaystyle H(x', t')}{\overset{\parallel}{H(x', t)}}$$ ---- by property (*) of $\sim$

$$\Rightarrow q(H(x,t)) = q(H(x', t'))$$

$\Rightarrow$ $q \circ H$ is constant on the fibers of $(q \times \mathbb{1})$.

$\Rightarrow$ $\exists$ well-defined, cts map $\bar{H} : Y \times I \to Y$

$$\boxed{\bar{H}(q(x), t) = q(H(x,t))}$$

Now Check that

- $\bar{H}(y, 0) = y$, $\forall y \in Y$
- $\bar{H}(y, 1) \in B$, $\forall y \in Y$
- $\bar{H}(b, t) = b$, $\forall t \in I$, $\forall b \in B$.
