MATH 5853-001 Topology I Fall 2010

Sample Questions

1. Give the definition of a *total order* on a set. Define the *order topology* on a totally ordered set. Define what it means for a topological space to be *Hausdorff.*

   (a) Prove that order topology is Hausdorff.

   (b) Prove that a space Z is Hausdorff iff the diagonal $\Delta_Z \subset Z \times Z$ is closed where $Z \times Z$ has the product topology.

   (c) Suppose that $f, g : X \to Y$ are continuous functions from a top space X to a totally ordered set Y with the order topology. Prove that $\{x \in X \mid f(x) \geq g(x)\}$ is closed in X.

2. State the Axiom of choice. Let $X_\alpha | \alpha \in J$ be an indexed collection of sets.

   (a) Define the product $\prod_{\alpha \in J} X_\alpha$.

   (b) Prove that the fact that the product of a nonempty collection of nonempty sets is nonempty is equivalent to the axiom of choice.

   (c) Define the product topology on $\prod_{\alpha \in J} X_\alpha$.

   (d) Prove that the projection maps $P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ are continuous.

   (e) Prove that a function $f : Z \to \prod_{\alpha \in J} X_\alpha$ is continuous iff $P_\alpha \circ f$ are continuous.

3. Define *quotient topology* and *quotient map.*

   Throughout this question we will call $\mathbb{R}/\mathbb{Z}$ the *circle* and $\mathbb{R}^2/\mathbb{Z}^2$ the *torus.*

   (a) Let $q : X \to Y$ be a quotient map, and let $f : X \to Z$ be a surjective function with the property that $f(x_1) = f(x_2)$ iff $q(x_1) = q(x_2)$. Prove that f induces a well defined bijection $\overline{f} : Y \to Z$ by $\overline{f}(q(x)) = f(x)$, and prove that $\overline{f}$ is continuous iff f is continuous.

   (b) Let $A \in SL(2, \mathbb{Z})$. Prove that A induces a homeomorphism $\mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$.

   (c) Check that the map $\mathbb{R} \to \mathbb{R}^2 : x \mapsto (x, 0)$ induces a continuous map from the circle $\mathbb{R}/\mathbb{Z}$ to the torus $\mathbb{R}^2/\mathbb{Z}^2$. The image is called the $(1, 0)$–curve on the torus.

   (d) Suppose that $p, q \in \mathbb{Z}$. Check that the map $\mathbb{R} \to \mathbb{R}^2 : x \mapsto (px, qx)$ induces a continuous map from the circle $\mathbb{R}/\mathbb{Z}$ to the torus $\mathbb{R}^2/\mathbb{Z}^2$. The image is called the (p, q)–curve on the torus.

   (e) Say that a subspace $A \subset X$ is a *retract* of X if there exists a continuous map $r : X \to A$ so that $r \circ i = \mathbb{I}_A$, where $i : A \to X : a \mapsto a$ is the inclusion map. Prove that the $(1, 0)$–curve is a retract of the torus $\mathbb{R}^2/\mathbb{Z}^2$.

   (f) Suppose $p, q \in \mathbb{Z}$ satisfy $\gcd(p, q) = 1$. Is the (p, q)–curve a retract of the torus $\mathbb{R}^2/Z^2$? Give reasons for your answer.

4. Define the *Mobius band*, M, to be the following quotient space of $[0, 10] \times [-1, 1]$.

$$M = [0, 10] \times [-1, 1]/ \sim$$

   where $\sim$ is defined by $(0, y) \sim (10, -y)$ for all $y \in [-1, 1]$.

   (a) Prove that $[0, 10] \times \{0\}/ \sim$ is homeomorphic to the circle $\mathbb{R}/\mathbb{Z}$. So we can safely refer to $[0, 10] \times \{0\}/ \sim$ as a *circle.*

   (b) Prove that the circle $[0, 10] \times \{0\}/ \sim$ is a retract of M. That is, construct a retract map $r : M \to [0, 10] \times \{0\}/ \sim$.

5. Define *closure* $\overline{A}$, *interior* A° for a subset A of a topological space X.

   Define the *boundary* (frontier), ∂A, of A as follows $\partial A = \overline{A} - A^\circ$.

   Let X and Y be topological spaces, $A \subset X$, $B \subset Y$, and give $X \times Y$ the product topology.

   (a) Prove that $\overline{A \times B} = \overline{A} \times \overline{B}$.

   (b) Prove that $(A \times B)^\circ = A^\circ \times B^\circ$.

   (c) Prove that $\partial(A \times B) = (\partial A \times B) \cup (A \times \partial B)$. Draw a picture in the case $A = B = [0,1]$ and $X = Y = \mathbb{R}$.

6. Let S_Ω denote a well-ordered set whose order type is the first uncountable ordinal. Give an argument to show that such a well-ordered set exists.

   Give $S_\Omega \cup \{\Omega\}$ the ordering in which every element of S_Ω is less than Ω, and consider the corresponding order topology.

   (a) Prove that Ω is a limit point of S_Ω.

   (b) Prove that every countable subset $A \subset S_\Omega$ has an upper bound in S_Ω.

   (c) Prove that no sequence in S_Ω converges to Ω.

   (d) Is $S_\Omega \cup \{\Omega\}$ with the order topology metrizable?

7. Is the projection $p_1 : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto x$ an open map? Is it a closed map?

8. Let $X = \{(x,y) \in \mathbb{R}^2 \mid y = 0 \text{ or } x \geq 0\}$. Is $p_1|_X$ an open map? Is it a closed map?