Prop: For every integer m, $3 \mid (m^3 - m)$

Pf By the Division Algorithm, given any integer m, there exist unique integers q and r so that

$$m = 3q + r, \quad \text{and} \quad 0 \leq r < 3.$$

This gives three possibilities for m; namely,

$m = 3q$ for some integer q,

$m = 3q + 1$ for some integer q, and

$m = 3q + 2$ for some integer q.

We deal with each of those cases separately.

CASE 1 $m = 3q$ for some $q \in \mathbb{Z}$

$$\Rightarrow m^3 - m = (3q)^3 - (3q) = 3(9q^3 - q)$$

which ~~is~~ divisible by 3.

CASE 2 $m = 3q + 1$ for some $q \in \mathbb{Z}$.

$$\Rightarrow m^3 - m = (3q+1)^3 - (3q+1)$$

$$= (3q)^3 + 3(3q)^2(1) + 3(3q)(1)^2 + \cancel{1} - 3q - \cancel{1}$$

$$= 3\left(9q^3 + (3q)^2 + (3q) - (3q)\right)$$

is divisible by 3.

Case 3 $\quad m = 3q + 2$ for some integer $q \in \mathbb{Z}$.

$$\Rightarrow m^3 - m = (3q+2)^3 - (3q+2)$$

$$= (3q)^3 + 3(3q)^2(2) + 3(3q)(2)^2 + 2^3 - 3q - 2$$

$$= 3\left(9q^3 + (3q)^2(2) + (3q)(2)^2 - 3q\right) + \underbrace{8 - 2}_{\substack{\shortparallel \\ 6 \,=\, 2(3)}}$$

$$= 3\left(9q^3 + (3q)^2(2) + (3q)(2)^2 - 3q + 2\right)$$

is divisible by 3.

In all cases $3 \mid (m^3 - m)$.

Therefore $3 \mid (m^3 - m) \quad \forall m \in \mathbb{Z}$

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