Monday 11/21/2005 Midterm III 10:30am–11:20am

Name: Student ID:

Instructions.

1. Attempt all questions.
2. Do not write on back of exam sheets. Extra paper is available if you need it.
3. Show all the steps of your work clearly.

Question	Points	Your Score
Q1	10	
Q2	10	
Q3	10	
Q4	10	
Q5	10	
TOTAL	50	

Q1]. . . [10 points] State the *Principle of Induction*.

Give a proof by induction that

$$\sum_{i=1}^{n} i^2 \;=\; \frac{n(n+1)(2n+1)}{6}$$

Q2]. . . [10 points] Give the definition of $a \equiv b \pmod{m}$.

Suppose that $a \equiv b \pmod{m}$, and that $a' \equiv b' \pmod{m}$. Prove **one** of the following conclusions. $a + a' \equiv b + b' \pmod{m}$, and $aa' \equiv bb' \pmod{m}$.

Find the remainder when 123^{456} is divided by 7. That is, compute $123^{456} \pmod{7}$.

Q3]. . . [10 points] State the *Schröder-Bernstein Theorem*.

Use the Schröder-Bernstein Theorem to prove one of the following (your choice).

- $\mathcal{P}(\mathbb{Z}^+)$ and $(0, 1)$ have the same cardinality.
- $(0, 1)$ and $(0, 1)^2$ have the same cardinality.

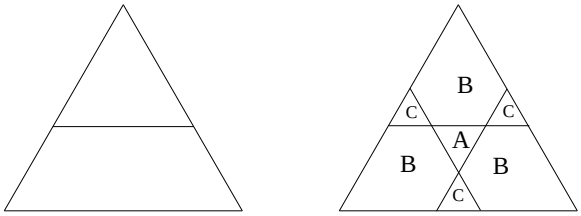
Q4]. . . [10 points] Give the definition of the greatest common divisor, $\gcd(a, b)$, of two integers a and b.

Compute $\gcd(180, 96)$ and show how to express your answer as an integer linear combination of 180 and 96.

Prove that if $a|bc$ and $\gcd(a, b) = 1$, then $a|c$.

Prove that if p is a prime number, and $p|ab$ for integers a and b, then $p|a$ or $p|b$.

Q5]. . . [10 points] Consider a pair of equilateral triangles such that the area of the larger is 3 times the area of the smaller. Take three copies of the smaller triangle inside the larger. A copy of the smaller triangle is based at each of the three vertices of the larger triangle. These overlap to form regions with area A, B and C as shown.



Show how to turn this into a proof by infinite descent (well-ordering) that $\sqrt{3}$ is irrational. Give a detailed algebra proof of the irrationality of $\sqrt{3}$ using infinite descent.