MATH 2423–Section 001 Calculus II Extra Homework I

Due on Thursday, February 8, 2001

In this homework we'll see how to compute some definite integrals without using the Fundamental Theorem of Calculus. You'll learn tricks and see that it may not always be best to subdivide an interval into equal width subintervals. But most of all, you should come away with an appreciation of the power and usefulness of the Fundamental Theorem of Calculus.

Background on Geometric Progressions. You should remember geometric progressions from your *compound interest* days in high school. A geometric progression is a sequence of numbers of the form

$$a,\ ar,\ ar^2, \ldots, ar^n$$

where $a, r \neq 0$ are numbers. The number r is usually called the common ratio.

For example, suppose you invest a principal P for n years at an annual interest rate of r percent. If the interest is compounded annually, then you have $P + \frac{r}{100}P = P(1 + \frac{r}{100})$ at the end of year 1, $P(1+\frac{r}{100}) + \frac{r}{100}P(1+\frac{r}{100}) = P(1+\frac{r}{100})^2$ at the end of year 2, and so on until you have $P(1+\frac{r}{100})^n$ at the end of year n. In this case we have a geometric progression with initial amount P and common ratio of $(1 + \frac{r}{100})$.

As a second example, the successive digits in the decimal number $0.33\ldots3$ stand for

$$\frac{3}{10}, \frac{3}{100}, \frac{3}{1000}, \cdots \frac{3}{10^n}$$

which form a geometric progression with common ratio equal to $\frac{1}{10}$.

There are instances where one wants to compute the sum of a geometric progression. In the first example above, if we were to keep investing P at the start of every year, then the balance in the account at the start of the $n+1$-st year would be the sum

$$P \ + \ P(1 + \frac{r}{100}) \ + \ \cdots \ + \ P(1 + \frac{r}{100})^n$$

In the second example, the value of the decimal $0.33\ldots3$ is precisely the sum

$$\frac{3}{10} + \frac{3}{100} + \cdots + \frac{3}{10^n}$$

Is there a simple expression for the following sum?

$$S \ = \ a + ar + \cdots + ar^n$$

Yes there is, and it's your first exercise to derive it.

Q1]... Prove that the sum of the geometric progression above is given by

$$S \ = \ \frac{a(r^{n+1} - 1)}{(r-1)}$$

[Hint: Write down the long sum rS explicitly, and put the sum S underneath it. Now subtract and tidy up!]

Q2]... What number does the following sum give?

$$1 + 1/2 + 1/4 + \cdots + 1/2^{10}$$

Computing $\int_a^b x^\alpha\, dx$ for rational $\alpha \neq -1$.

We shall assume throughout that $0 < a < b$ for simplicity. We warm up to the general result by proving this for $\alpha > 0$ an integer.

Use a geometric progression to choose a partition of $[a, b]$ into n subintervals. That is let $q = \sqrt[n]{b/a}$ and consider the following partition

$$a,\ aq,\ aq^2,\ \dots\ aq^n = b$$

Q3]... This has several steps.

- Using left hand endpoints as evaluation points, verify that the Riemann sum is

$$a^{\alpha+1}(q-1)\{1 + q^{\alpha+1} + q^{2(\alpha+1)} + \cdots + q^{(n-1)(\alpha+1)}\}$$

- Show that this sum is just given by

$$a^{\alpha+1}(q-1)\frac{q^{n(\alpha+1)} - 1}{q^{\alpha+1} - 1}$$

- Now, remembering that $q^n = b/a$, rewrite the expression above as

$$(b^{\alpha+1} - a^{\alpha+1})\frac{q-1}{q^{\alpha+1} - 1}$$

- Then note that the fraction part above is just the reciprocal of $1 + q + \cdots + q^\alpha$ (why?).

- What happens to $q = (b/a)^{1/n}$ as $n \to \infty$? Complete the derivation of the limit of the Riemann sum as $n \to \infty$. Compare your answer with the result that anti-differentiation would give (Fund Theorem).

Now we're ready for the case of rational $\alpha \neq -1$.

Q4]... Again there are several steps.

- Verify that everything proceeds exactly as above, and that we're left with the same problem of evaluating the limit of

$$\frac{q-1}{q^{\alpha+1} - 1}$$

  as $n \to \infty$ (or, equivalently, as $q \to 1$).

- Suppose that $\alpha > 0$. Let $\alpha = r/s$ for positive integers r and s. Write $q^{1/s} = t$. Note that $t \to 1$ as $q \to 1$, so we can reduce to (why?) computing the limit of

$$\frac{t^s - 1}{t^{r+s} - 1}$$

  as $t \to 1$.

- Divide above and below by $(t-1)$ and use sums of geometric progressions to recognize this fraction as

$$\frac{t^{s-1} + \cdots + t + 1}{t^{r+s-1} + \cdots + t + 1}$$

  which tends to $s/(r+s) = 1/(\alpha+1)$ as $t \to 1$.

- Finally, we consider the case where $\alpha < 0$ ($\alpha \neq -1$). In this case, set $q^{-1/s} = t$ and proceed as above (show details).

Computing $\int_a^b \sin x \, dx$ and $\int_a^b \cos x \, dx$.

In this example we can use equal width subintervals, but have to use a slick telescoping sum trick together with some trig identities and a famous trig limit.

Q5]... Here are the steps for the $\sin x$ integral.

- Let $h = (b - a)/n$. Using right-hand endpoints, show that the Riemann sum is

$$h\{\sin(a+h) + \sin(a+2h) + \cdots + \sin(a+nh)\}$$

- So long as h is not a multiple of 2π we can multiply and divide by $2\sin(h/2)$, and use the trig formula

$$2\sin A \sin B \;=\; \cos(A-B) - \cos(A+B)$$

  to get a telescoping sum of cosines. They cancel (all but two to give)

$$\frac{h}{2\sin(h/2)}\{\cos(a+h/2) - \cos(b+h/2)\}$$

  Fill in the details!

- Use the $\frac{\sin\theta}{\theta}$ limit to determine the limit of these Riemann sums as $n \to \infty$ ($h \to 0$). Compare your answer with the result guaranteed by the Fund Theorem.

Q6]... Compute the cos definite integral using a limit of Riemann sums analogous to the one outlined above. Compare your answer to the one obtained by using the Fund Theorem.