Homework # 6

Section 6.4 : Work

1) $W = F.d = 900.8 = 7200\,J.$

2) $F = m.g = 60.(9.8) = 588N$; $W = F.d = 588.2 = 1176\,J.$

8) $f(x) = 25$ and $f(x) = k.x = k.(0.1)$ hence $k.(0.1) = 25 \Rightarrow k = 250\,N/m$

$10\,cm = 0.1\,m$

and so $f(x) = 250x$

Now, since $5\,cm = 0.05\,m$,

$$W = \int_0^{0.05} 250x\,dx = \left[\frac{250x^2}{2}\right]_0^{0.05} \approx 0.3\,J.$$

9) If $W = \int_0^{0.12} kx\,dx = 2.J$ then $2 = \left[\frac{1}{2}kx^2\right]_0^{0.12} \Rightarrow \frac{1}{2}k.(0.12)^2 = 2$

$\Rightarrow k = \frac{2500}{9}\,N/m$

hence the work needed to stretch from $35\,cm = 0.35\,m$ to $40\,cm = 0.4\,m$

is

$$\int_{0.05}^{0.1} \frac{2500}{9}x\,dx = \left[\frac{1250}{9}x^2\right]_{1/20}^{1/10} = \frac{25}{24}\,J.$$

10) If $12 = \int_0^1 kx\,dx = \left[\frac{1}{2}kx^2\right]_0^1 = \frac{1}{2}k$ then $k = 24\,lb/ft$. and the work

required is $\int_0^{3/4} 24x\,dx = \left[12x^2\right]_0^{3/4} = \frac{27}{4}\,ft\text{-}lb.$

13) a) The portion of the rope from x ft to $(x+\Delta x)$ ft below the top of the building weighs $\frac{1}{2}\Delta x$ lb and must be lifted x_i^* ft, so its contribution to the total work is $\frac{1}{2}x_i^*\Delta x$ ft-lb. The total work is

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}x_i^*\Delta x = \int_0^{50}\frac{1}{2}x\,dx = 625 \text{ ft-lb.}$$

b) When half the rope is pulled to the top of the building, the work to lift the top half of the rope is

$$W_1 = \int_0^{25}\frac{1}{2}x\,dx = \frac{625}{4}\text{ ft-lb.}$$

The bottom half of the rope is lifted 25 ft and the work needed to accomplish that is

$$W_2 = \int_{25}^{50}\frac{1}{2}\cdot 25\,dx = \frac{25}{2}[x]_{25}^{50} = \frac{625}{2}\text{ ft-lb.}$$

The total work done in pulling half the rope to the top of the building is

$$W = W_1 + W_2 = \frac{625}{2}+\frac{625}{4} = \frac{3}{4}\cdot 625 = \frac{1875}{4}\text{ ft-lb.}$$

14) Note: 1) After lifting, the chain is L-shaped, with 4 m of the chain lying along the ground

2) The chain slides effortlessly and without friction along the ground while its end is lifted.

3) The weight density of the chain is constant throughout its length and therefore equals $(8\ kg/m).(9.8\ m/s^2) = 78.4\ N/m$.

The part of the chain $x \cdot m$ from the lifted end is raised $6-x$ m if $0 \leq x \leq 6$ m, and it is lifted 0 m if $x > 6$m. Thus

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} (6-x_i^*) . 78.4 \Delta x = \int_0^6 (6-x). 78.4\,dx = \int_0^6 6.78.4\,dx - \int_0^6 78.4\,x\,dx$$
$$= (78.4).(18).$$

16) The work needed to lift the bucket itself is $4.80 = 320$.
At time t (in seconds) the bucket is $x_i^* = 2t$ ft above its original 80 ft depth, but it now holds only $(40-0.2t)$ lb of water. In terms of distance, the bucket holds $\left[40 - 0.2.\frac{1}{2}x_i^*\right]$ lb of water when it is x_i^* ft above its original 80 ft depth. Moving this amount of water a distance Δx requires $\left(40 - \frac{1}{10}x_i^*\right).\Delta x$ ft-lb of work. Thus the work needed to lift the water is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \left(40 - \frac{1}{10}x_i^*\right)\Delta x = \int_0^{80} \left(40 - \frac{1}{10}x\right)dx = (3200-320)\ ft\text{-}lb$$

Adding the work of lifting the bucket gives a total of 3200 ft-lb of work.

29) $W = \int_0^b F(r)dr = \int_0^b G\frac{m_1.m_2}{r^2}dr = G.m_1.m_2\left[\frac{-1}{r}\right]_a^b = G\,m_1 m_2\left[\frac{1}{a} - \frac{1}{b}\right]$

30) By (29), $W = G.M.m\left(\frac{1}{R} - \frac{1}{R+1{,}000{,}000}\right)$ where M = Mass of earth in kg, R = radius of earth in m and m = mass of satellite in kg. Thus

$$W = (6.67\times10^{-11}).(5.98\times10^{24}).(1000).\left(\frac{1}{6.37\times10^6} - \frac{1}{7.37\times10^6}\right) \approx 8.50\times10^9\ J$$

Section 6.5: Average Value of a Function

1) $f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{1-(-1)}\int_{-1}^{1} x^2 dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$

2) $f_{ave} = \frac{1}{2-0}\int_0^2 (x-x^2)dx = \frac{1}{2}\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^2 = -\frac{1}{3}$

3) $g_{ave} = \frac{1}{\frac{\pi}{2}-0}\int_0^{\pi/2}\cos x dx = \frac{2}{\pi}\left[\sin x\right]_0^{\pi/2} = \frac{2}{\pi}$

20) $s = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{2s/g}$ (since $t \geq 0$)

Now $v = \frac{ds}{dt} = g \cdot t = g\sqrt{2s/g} = \sqrt{2gs} \Rightarrow v^2 = 2gs \Rightarrow s = \frac{v^2}{2g}$. So

$$v = F(t) = g \cdot t$$

and

$$v = G(s) = \sqrt{2gs}$$

Note that

$$v_T = F(T) = gT$$

Displacement can be viewed as a function of t: $s = s(t) = \frac{1}{2}gt^2$; also

$$s(t) = \frac{v^2}{2g} = \frac{[F(t)]^2}{2g}$$

when $t = T$, these two formulas imply that

$$\sqrt{2g\,s(T)} = F(T) = v_T = gT = \frac{2\left(\frac{1}{2}gT^2\right)}{T} = \frac{2 \cdot s(t)}{T}$$

The average of the velocities with respect to time t during the interval $[0,T]$ is

$$v_{t\text{-ave}} = F_{ave} = \frac{1}{T-0}\int_0^T F(t)\,dt \overset{\text{by F.T.C.}}{=} \frac{1}{T}\left[s(T)-s(0)\right]$$

$$= \frac{s(T)}{T} = \frac{1}{2}v_T \quad (\text{since } s(0)=0)$$

But the average of the velocities with respect to displacement s during the corresponding displacement interval $[s(0), s(T)] = [0, s(T)]$ is

$$v_{s\text{-ave}} = G_{ave} = \frac{1}{s(T)-0}\int_0^{s(T)} G(s)\,ds = \frac{1}{s(T)}\int_0^{s(T)} \sqrt{2gs}\,ds = \frac{\sqrt{2g}}{s(T)}\int_0^{s(T)} s^{1/2}\,ds$$

$$= \frac{\sqrt{2g}}{s(T)}\,\frac{2}{3}\left[s^{3/2}\right]_0^{s(T)} = \frac{2}{3}\,\frac{\sqrt{2g}}{s(T)}\,[s(T)]^{3/2} = \frac{2}{3}\sqrt{2gs(T)}$$

$$= \frac{2}{3}v_T$$

23) Let $F(x) = \int_a^x f(t)\,dt$ for $x \in [a,b]$. Then F is continuous on $[a,b]$ and differentiable on (a,b), so by the mean value theorem, there is a number c in (a,b) such that $F(b)-F(a) = F'(c).(b-a)$. But $F'(x) = f(x)$ by the F.T.C. therefore

$$\int_a^b f(t)\,dt - 0 = f(c).(b-a)$$

24) $$f_{ave}[a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^c f(x)\,dx + \frac{1}{b-a}\int_c^b f(x)\,dx$$

$$= \frac{c-a}{b-a}\left[\frac{1}{c-a}\int_a^c f(x)\,dx\right] + \frac{b-c}{b-a}\left[\frac{1}{b-c}\int_c^b f(x)\,dx\right]$$

$$= \frac{c-a}{b-a}\cdot f_{ave}[a,c] + \frac{b-c}{b-a} f_{ave}[c,b].$$