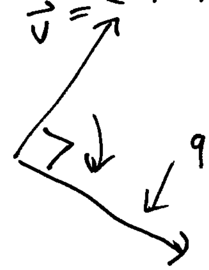
Extra Hwk III soln to Q1.

Fact ① [Rotating vector 90° clockwise]

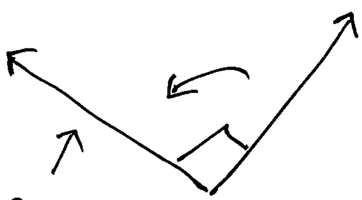
$\vec{v} = \langle c, d \rangle$



90° clockwise rotation of $\vec{v} = \langle d, -c \rangle$

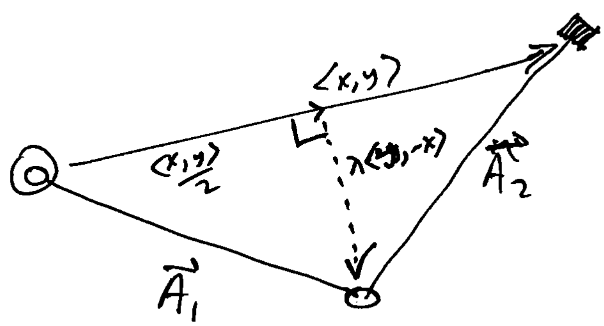
Fact ② [Rotating vector 90° counter clockwise]

$\vec{v} = \langle c, d \rangle$



90° counterclockwise rotation of $\vec{v} = \langle -d, c \rangle$

$\langle x, y \rangle$ = position vector of end of planimeter



$\vec{A}_1 = \langle a, b \rangle$ = "1st arm" vector of planimeter.

$\vec{A}_2$ = "2nd arm" vector of planimeter.

Length of arms.

$$\boxed{\vec{A}_2 = \langle x, y \rangle - \vec{A}_1} \qquad \boxed{|\vec{A}_1| = |\vec{A}_2| = L}$$

Step 1 Figure out expression for $\vec{A}_1 = \langle a, b \rangle$.

From 1st Figure (previous page) we see ...

$$\langle a,b \rangle = \tfrac{1}{2}\langle x, y\rangle + \left(\text{vector turned } 90^\circ \text{ clockwise from } \langle x,y\rangle\right)$$

$$= \tfrac{1}{2}\langle x,y\rangle + \lambda\langle y,-x\rangle \quad \text{--- by Fact 1}$$

$$= \left\langle \frac{x}{2} + \lambda y , \frac{y}{2} - \lambda x \right\rangle$$

Now $|\langle a,b\rangle| = L \Rightarrow$

$$\frac{x^2}{4} + \lambda^2 y^2 + \cancel{2\frac{\lambda}{2}xy} + \frac{y^2}{4} + \lambda^2 x^2 - \cancel{2\frac{\lambda}{2}xy} = L^2$$

$$\lambda^2(x^2+y^2) + \frac{x^2+y^2}{4} = L^2$$

$$(x^2+y^2)\lambda^2 = L^2 - \frac{(x^2+y^2)}{4} = \frac{4L^2-(x^2+y^2)}{4}$$

$$\boxed{\lambda = \frac{\sqrt{4L^2-(x^2+y^2)}}{2\sqrt{(x^2+y^2)}}}$$

$$\vec{A}_1 = \langle a,b\rangle = \left\langle \frac{x}{2} + \frac{y}{2}\frac{\sqrt{4L^2-(\ )}}{\sqrt{(\ )}} , \frac{y}{2} - \frac{x}{2}\frac{\sqrt{4L^2-(\ )}}{\sqrt{(\ )}} \right\rangle$$

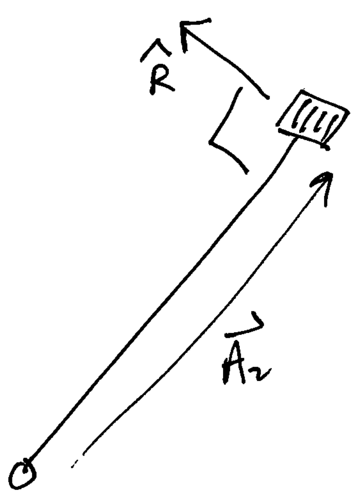
$$\vec{A}_2 = \langle x, y \rangle - \vec{A}_1$$

$$\vec{A}_2 = \left\langle \frac{x}{2} - \frac{y}{2}\frac{\sqrt{4L^2 - (\ )}}{\sqrt{(\ )}} \,,\; \frac{y}{2} + \frac{x}{2}\frac{\sqrt{4L^2 - (\ )}}{\sqrt{(\ )}} \right\rangle$$

<u>exercise</u>: check that $|\vec{A}_2|^2 = \left(\frac{x}{2} - \frac{y}{2}\frac{\sqrt{\ }}{\sqrt{\ }}\right)^2 + \left(\frac{y}{2} + \frac{x}{2}\frac{\sqrt{\ }}{\sqrt{\ }}\right)^2$

$= - - - - - = L^2$!

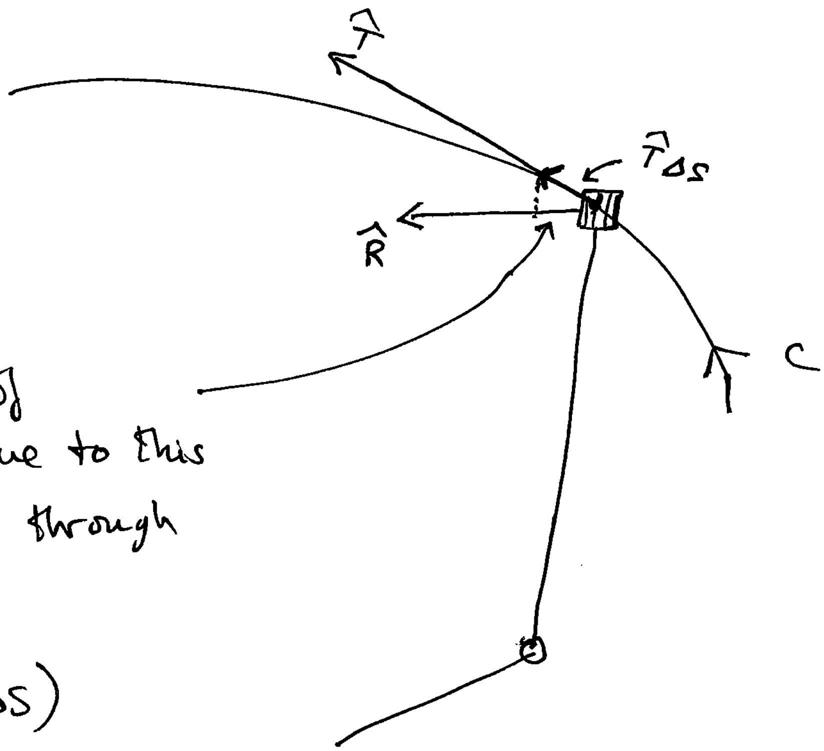
Rolling Vector $\hat{R}$ = unit vector obtained from $\vec{A}_2$ by rotation through 90° counterclockwise



Fact ② ⇒

$$\boxed{\hat{R} = \frac{1}{2L}\left\langle -y - x\frac{\sqrt{4L^2 - (\ )}}{\sqrt{(\ )}} \,,\; x - y\frac{\sqrt{4L^2 - (\ )}}{\sqrt{(\ )}} \right\rangle}$$

<u>Now</u> when we move head of planimeter over a small segment Δs of curve C we can approximate this by a shift through $\hat{T}\,\Delta s$ where $\hat{T}$ = unit tangent vector.



amount of rolling due to this movement through $\hat{T}\,\Delta s$ is $\hat{R}\cdot(\hat{T}\,\Delta s)$

(Note ... amount of slipping is $\left(\dfrac{\vec{A}_2}{L}\right)\cdot(\hat{T}\,\Delta s)$)

Total amount of Rolling as we move head of planimeter all around C is then given as a Limit of Riemann sums ----

$$\oint_C \hat{R}\cdot\hat{T}\,ds$$

Finally, ...

$$\oint_C \hat{R}\cdot\hat{T}\,ds = \oint_C \hat{R}\cdot d\vec{r}$$

Green's Th$^{\underline{m}}$

$$= \iint_{\substack{\text{Region} \\ \text{enclosed by } C}} \text{curl}(\hat{R})\cdot\hat{k}\,dA \qquad -(**)$$

$$\text{Curl}(\hat{R}) = \frac{1}{2L}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y - x\frac{\sqrt{4L^2-(\ )}}{\sqrt{(\ \ )}} & x - y\frac{\sqrt{4L^2-(\ )}}{\sqrt{(\ \ )}} & 0 \end{vmatrix}$$

$$= \frac{1}{2L}\left\langle 0,\ 0,\ \frac{\partial}{\partial x}\left(x - y\frac{\sqrt{\ \ }}{\sqrt{\ \ }}\right) + \frac{\partial}{\partial y}\left(y + x\frac{\sqrt{\ \ }}{\sqrt{\ \ }}\right)\right\rangle$$

$$= \frac{1}{2L}\left\langle 0, 0, \underset{\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}}{2} \underbrace{- y\frac{\partial}{\partial x}\left(\frac{\sqrt{4L^2-(x^2+y^2)}}{\sqrt{x^2+y^2}}\right) + x\frac{\partial}{\partial y}\frac{\sqrt{4L^2-(x^2+y^2)}}{\sqrt{x^2+y^2}}}_{\text{These cancel!}}\right\rangle$$

$$= \langle 0, 0, \tfrac{1}{L} \rangle$$

Finally (*+) (Green's Thm) gives

Total amount of "Rolling" of planimeter wheel

$$= \oint_C (\vec{R} \cdot \hat{T})\, ds = \iint_{\substack{\text{Region} \\ \text{enclosed} \\ \text{by } C}} \langle 0, 0, \tfrac{1}{L} \rangle \cdot \hat{k}\, dA$$

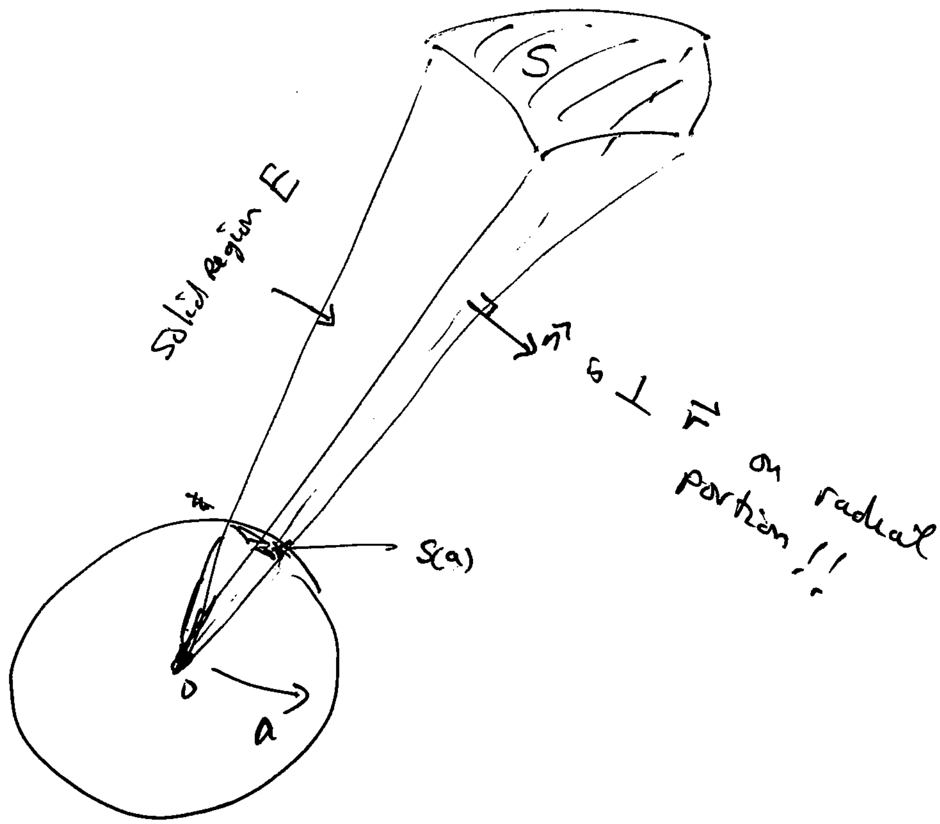
$$= \frac{1}{L} \iint_{\substack{\text{Region} \\ \text{enclosed} \\ \text{by } C}} dA$$

$$= \frac{1}{L} \text{Area}\left(\text{Region enclosed by } C\right)$$

This is what we wanted to show!

Measure of solid angle

$$|\Omega(S)| = \frac{\text{area }(S(a))}{a^2}$$



To show:

$$\boxed{|\Omega(S)| = \iint_S \frac{\vec{r}}{|\vec{r}|^3} \cdot \hat{n} \, dS} \quad \text{— [+]}$$

Let E = solid region obtained by coning S off to origin minus portion inside ball of radius a.

∂E is composed of 3 surfaces ...

$$\partial E = S \cup S(a) \cup \text{(Radial piece)}. \quad \text{— [++]}$$

Divergence Th$^{\underline{m}}$ $\Rightarrow$

$$\iint_{\partial E} \left(\frac{\vec{r}}{|\vec{r}|^3} \cdot \hat{n} \right) dS = \iiint_E \operatorname{div}\left(\frac{\vec{r}}{|\vec{r}|^3} \right) dV \quad \text{— [+++]}$$

Now
$$\operatorname{div}\left(\frac{\vec{r}}{|r|^3}\right) = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{\cancel{2}} \frac{x(\cancel{2}x)}{(x^2+y^2+z^2)^{5/2}}$$
$$+ \frac{1}{(\ )^{3/2}} - \frac{3}{\cancel{2}} \frac{y(\cancel{2}y)}{(\ )^{5/2}}$$
$$+ \frac{1}{(\ )^{3/2}} - \frac{3}{\cancel{2}} \cdot \frac{z(\cancel{2}z)}{(\ )^{5/2}}$$

$$= \frac{3}{(\ )^{3/2}} - \frac{3(x^2+y^2+z^2)}{(\ )^{5/2}}$$

$$= 0 \ !!$$

Thus [+++] becomes

$$\iint_{\partial E} \left(\frac{\vec{r}}{|\vec{r}|^3} \cdot \hat{n} \right) dS = \iiint_E 0 \, dv = 0 .$$

But [++] tells us that we can split LHS as a sum of 3 terms

$$\iint_S \left(\frac{\vec{r}}{|\vec{r}|^3} \cdot \hat{n} \right) dS - \iint_{S(a)} \left(\frac{\vec{r}}{|\vec{r}|^3} \cdot \hat{n} \right) dS + \underbrace{\cancel{\iint_{\substack{\text{Radial} \\ \text{piece}}} \left(\frac{\vec{r}}{|\vec{r}|^3} \cdot \hat{n} \right) dS}}_{=0}$$

outward pointing unit normal from E is negative of usual normal (outward pointing from ball of radius a).

Last term $= 0$ since $\frac{\vec{r}}{|\vec{r}|^3}$ is radial

& $\hat{n} \perp$ to $\vec{r}$ on the radial pieces!

$$\Rightarrow \iint_S \left(\frac{\vec{r}}{|\vec{r}|^3} \cdot \hat{n} \right) dS - \iint_{S(a)} \left(\frac{\vec{r}}{|\vec{r}|^3} \cdot \hat{n} \right) dS = 0$$

Thus

$$\iint_S \left(\frac{\vec{r}}{|\vec{r}|^3} \cdot \hat{n} \right) ds = \iint_{S(a)} \left(\frac{\vec{r}}{|\vec{r}|^3} \cdot \hat{n} \right) ds'$$

On sphere of radius a we know $\hat{n} = \frac{\vec{r}}{a}$

Also $|r| = a$.

$\Rightarrow$ RHS becomes

$$\iint_{S(a)} \frac{\vec{r}}{a^3} \cdot \frac{\vec{r}}{a} \, ds'$$

$$= \iint_{S(a)} \frac{|r|^2}{a^4} \, ds'$$

$$= \iint_{S(a)} \frac{a^2}{a^4} \, ds' = \frac{1}{a^2} \iint_{S(a)} ds'$$

$$= \frac{1}{a^2} \left(\text{Area of } S(a) \right)$$

Thus [+] holds!