Q1]... [14 points] Write down the definition of a *group*, and give an example.

A group $(G, \bullet)$ is a set together with a binary operation $\bullet : G \times G \to G : (g_1, g_2) \mapsto g_1 \bullet g_2$

satisfying

(i) $\bullet$ is associative:

$$(g_1 \bullet g_2) \bullet g_3 = g_1 \bullet (g_2 \bullet g_3) \quad \text{for all } g_1, g_2, g_3 \in G$$

(ii) existence of an identity:

$\exists\, 1 \in G$ such that $1 \bullet g = g \bullet 1 = g$, $\forall g \in G$,

(iii) existence of inverses:

$\forall g \in G$, $\exists\, g^{-1} \in G$ such that $g \bullet g^{-1} = g^{-1} \bullet g = 1$

Write down the definition of a *homomorphism*, and give an example.

A homomorphism is a function $\varphi : G_1 \to G_2$ where G_1, G_2 are groups, which satisfies

$$\varphi(g_1 g_2) = \varphi(g_1)\,\varphi(g_2) \quad \forall\, g_1, g_2 \in G_1.$$

Write down the definition of an *isomoprhism*, and give an example.

An isomorphism of groups G_1, G_2 is a <u>bijective homomorphism</u> $\varphi : G_1 \to G_2$.

[Q3 deals with an example of an iso, which is in turn an example of a hom.]

[Q4. proves that conjugation C_g is always an isomorphism : $G \to G$]

Q2]... [24 points] For each of the following examples, say whether the given set is a *group* under the given operation. If it is not a group, then give a reason why not.

(1) The integers $\mathbb{Z}$ under addition.

YES

(2) The integers $\mathbb{Z}$ under multiplication.

NO $\quad \nexists\, 0^{-1}$ (0 has no inverse)

(3) The non-negative rationals $\mathbb{Q}_{\geq 0}$ under multiplication.

No (0 has no inverse)

(4) The non-zero complex numbers under multiplication.

YES

(5) The set of injective maps of $\mathbb{Z} \to \mathbb{Z}$ under composition.

NO — injective maps which are NOT also surjections will not have inverses.

(6) The set of 2×2 matrices with real entries under addition.

YES

(7) The set of 2×2 matrices with real entries under matrix multiplication.

NO — problem with inverses $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ has det=0 $\Rightarrow$ no inverse

(8) The set $\{f \mid f : \mathbb{R} \to \mathbb{R} : x \mapsto ax^2 + b,\ a, b \in \mathbb{R}, a \neq 0\}$ under composition.

No $\to$ [parabola $\Rightarrow$ not injective $\Rightarrow$ not bijection $\Rightarrow$ No inverses

$\to$ No because composition gives x^4 functions $(x^2)^2 = x^4 \neq x^2$

Q3]... **[12 points]** Consider the following two groups.

$$G_1 = \{f \mid f : \mathbb{R} \to \mathbb{R} : x \to ax + b,\ a, b \in \mathbb{R},\ a \neq 0\}$$

under composition of maps, and

$$G_2 = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{R},\ a \neq 0\right\}$$

under multiplication of matrices.

Prove that these two groups are isomorphic. That is, write down an explicit isomorphim $\varphi : G_1 \to G_2$ and verify that it is an isomorphism.

$$\varphi : G_1 \longrightarrow G_2$$
$$: f \longmapsto \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$
$f(x) = ax+b$

φ is injective : $\varphi(ax+b) = \varphi(cx+d) \Rightarrow \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$

$\Rightarrow a = c \ \& \ b = d$

$\Rightarrow x \mapsto ax+b \ \& \ x \mapsto cx+d$ are identical functions.

φ is surjective : Given $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, the function

$f(x) = ax+b$ is in G_1

$\& \ \varphi(f') = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \Rightarrow$ surjective!

φ is a homomorphism

Let $\boxed{f(x) = ax+b}$ $\boxed{g(x) = cx+d}$

then $f \circ g(x) = f(g(x)) = f(cx+d) = a(cx+d) + b$

$\Rightarrow \boxed{(f \circ g)(x) = \underline{ac}x + (\underline{ad+b})}$

$\varphi(f) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$

$\varphi(g) = \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$

$\varphi(f \circ g) = \begin{pmatrix} ac & ad+b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \varphi(f)\varphi(g)$

Q4]... [14 points] Let G be a group and $g \in G$. Prove that the conjugation map $C_g : G \to G : x \mapsto gxg^{-1}$ is a bijective map.

$C_g(x) = C_g(y)$

$\Rightarrow gxg^{-1} = gyg^{-1}$

$\Rightarrow xg^{-1} = yg^{-1}$ ... left cancellation

$\Rightarrow x = y$ --- right cancellation

$\Rightarrow C_g$ is injective — (1)

Given $x \in G$, note that $g^{-1}xg \in G$ also, and

$$C_g(g^{-1}xg) = g(g^{-1}xg)g^{-1} = gg^{-1}xg^{-1}g = x$$

So C_g is SURJECTIVE — (2)

(1),(2) $\Rightarrow$ C_g is a bijection.

When is C_g a group homomorphism? $\rightsquigarrow$ Does $C_g(xy) = C_g(x)\,C_g(y)$?

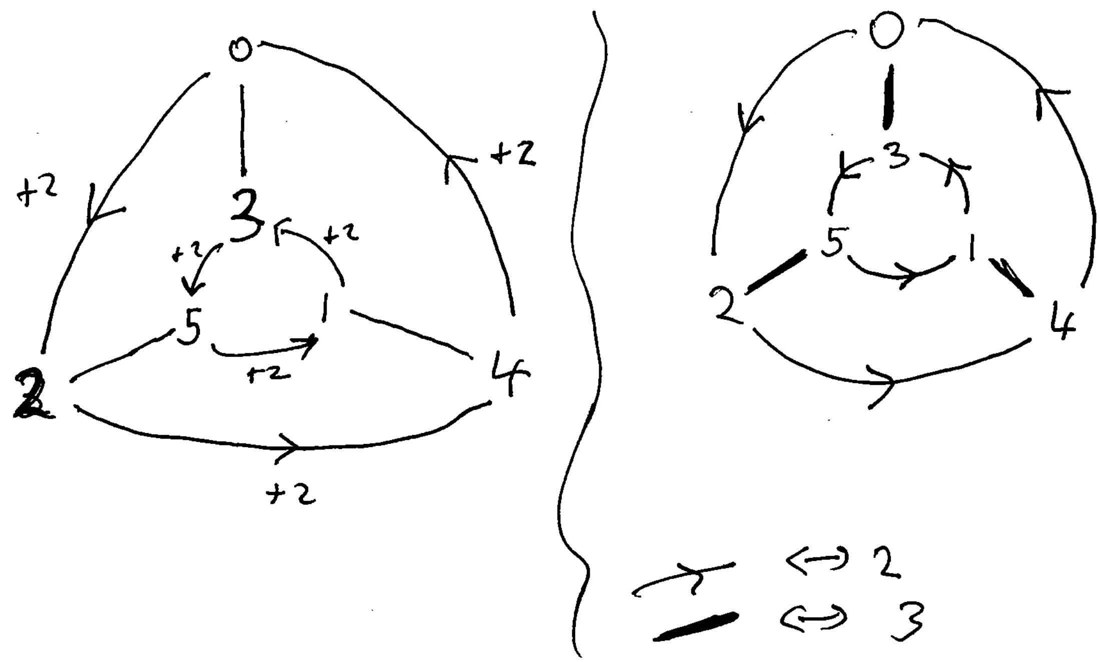
$C_g(xy) = g(xy)g^{-1}$

$C_g(x)\,C_g(y)$
$= (gxg^{-1})(gyg^{-1})$
$= gx\cancel{g^{-1}}\cancel{g}yg^{-1}$
$= gxyg^{-1}$

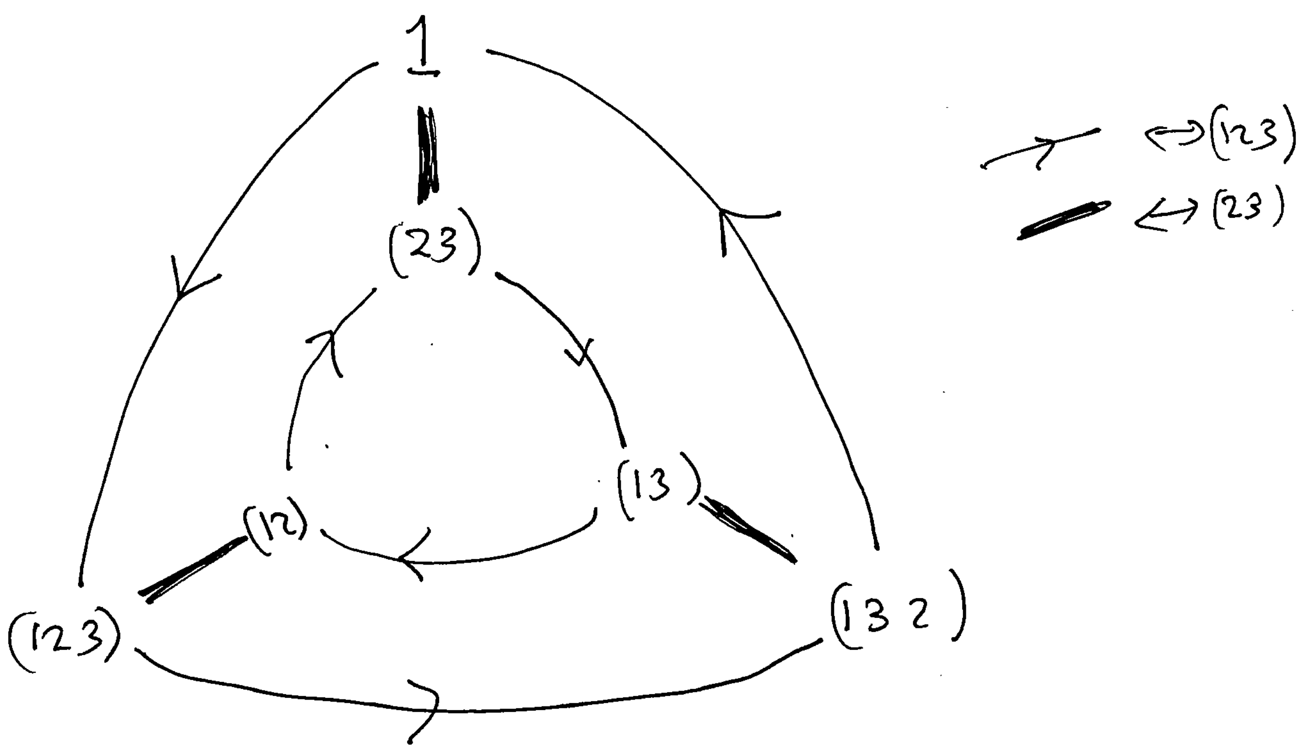
SAME

$\Rightarrow C_g$ is (always) a homomorphism!

Q5]... **[20 points]** Draw the Cayley graph of $\mathbb{Z}_6$ with respect to the generating set $\{2,3\}$.



Draw the Cayley graph of the symmetric group S_3 with respect to the generating set $\{(23),(123)\}$.



Q6]... [16 points] Compute the following product of permutations.

$$(1234)(2315)(1234)^{-1}$$

$$= (3425)$$

Find 5 different elements g of S_5 which satisfy the equation

$$g(12345)g^{-1} = (13452)$$

Note ... $g(12345)g^{-1} = (g(1)\,g(2)\,g(3)\,g(4)\,g(5))$ so there are 5 cases!

$(g(1)\,g(2)\,g(3)\,g(4)\,g(5)) = (13452)$

$g(1)=1$, $g(2)=3$, $g(3)=4$, $g(4)=5$, $g(5)=2$

$\boxed{g=(2345)}$

$(g(1)\,g(2)\,g(3)\,g(4)\,g(5)) = (21345)$

$g(1)=2$, $g(2)=1$, $g(3)=3$, $g(4)=4$, $g(5)=5$

$\boxed{g=(12)}$

$(g(1)\,g(2)\,g(3)\,g(4)\,g(5)) = (52134)$

$g(1)=5$, $g(2)=2$, $g(3)=1$, $g(4)=3$, $g(5)=4$

$\boxed{g=(1543)}$

$(g(1)\,g(2)\,g(3)\,g(4)\,g(5)) = (45213)$

$\boxed{g=(14)(253)}$

$g(1)=4$, $g(2)=5$, $g(3)=2$, $g(4)=1$, $g(5)=3$

$(g(1)\,g(2)\,g(3)\,g(4)\,g(5)) = (34521)$

$\boxed{g=(135)(24)}$

$g(1)=3$, $g(2)=4$, $g(3)=5$, $g(4)=2$, $g(5)=1$