Q1]... **[20 points]** Say whether each of the following statements is True or False.

(1) The equation $\det(AB) = \det(BA)$ holds for all $n \times n$ matrices A and B.

TRUE (both are equal to $\det(A)\det(B)$)

(2) If A is a 3×3 matrix, then $\det(2A) = 2\det(A)$.

FALSE $\left(\det(2A) = 2^3 \det(A) = 8\det(A) \right)$

(3) The collection of all 2×2 matrices A such that $\det(A) = 0$ is a subspace of the vector space $M_{2\times 2}$.

False $\left(\det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 = \det\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{yet } \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det(I_2) = 1 \neq 0. \right)$

(4) If $\{v_1, v_2\}$ is a linearly independent collection of vectors in a vector space V, then so is the collection $\{v_1, v_2, 3v_1 + 2v_2\}$.

False (the third vector is a nontrivial lin. comb of the other 2)

(5) If the collection $\{v_1, v_2\}$ spans the vector space V, then so does the collection $\{v_1, v_2, 3v_1 + 2v_2\}$.

True. (can always add $0(3v_1 + 2v_2)$ to any given sum).

Q2]...[20 points] Compute the determinant of the following matrix:

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= 2 \det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = 2 \cdot \det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} = 2\left((1)(1)(-2)\right) = \boxed{-4.}$$

Find the area of the triangle in $\mathbb{R}^2$ with vertices $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

difference vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

and $\begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$

$$\text{Area} = \frac{1}{2}\left|\det\begin{pmatrix} -1 & 1 \\ 3 & -6 \end{pmatrix}\right| = \frac{1}{2}(6-3) = \frac{3}{2}$$

Q3]...[20 points] Let A be an $m \times n$ matrix.

(1) Define the null space Null(A) of A.

$$\mathrm{Null}(A) = \{ \vec{v} \in \mathbb{R}^n \mid A\vec{v} = \vec{0} \}$$ or in words, ...

Null(A) is the collection of all vectors $\vec{v}$ in $\mathbb{R}^n$ such that $A\vec{v} = \vec{0}$.

(2) Prove that Null(A) is a subspace of $\mathbb{R}^n$.

① $\vec{0} \in \mathbb{R}^n$ $\quad A\vec{0} = \vec{0} \quad \Rightarrow \quad \vec{0} \in \mathrm{Null}(A)$. Null(A) is not empty.

② if $\vec{v}_1, \vec{v}_2 \in \mathrm{Null}(A)$, then $A\vec{v}_1 = \vec{0} = A\vec{v}_2$. And so

$$A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = \vec{0} + \vec{0} = \vec{0}$$

$\Rightarrow \vec{v}_1 + \vec{v}_2 \in \mathrm{Null}(A)$ $\quad \Rightarrow$ closed under addition

③ If $\vec{v}_1 \in \mathrm{Null}(A)$ and $\lambda \in \mathbb{R}$, then

$$A(\lambda \vec{v}_1) = \lambda A\vec{v}_1 = \lambda \vec{0} = \vec{0} \quad \Rightarrow \lambda \vec{v}_1 \in \mathrm{Null}(A)$$

$\Rightarrow$ closed under scalar mult^n.

①②③ $\Rightarrow$ Null(A) is a subspace

(3) Give a geometric description of the subspace $\mathrm{Null}\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ of $\mathbb{R}^3$.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{v} \in \mathrm{Null}(A)$ means $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x + z = 0 \quad \Rightarrow \quad z = -x$

$2x + y + z = 0$

$x + y = 0 \quad y = -x$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -x \\ -x \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\mathrm{Null}(A) = \left\{ \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \;\middle|\; \lambda \in \mathbb{R} \right\}$$ = line through $\vec{0}$ in $\mathbb{R}^3$ in the direction of the vector $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

Q4]... **[20 points]** Show that the following collection of functions is linearly independent in the vector space of all real valued functions of a real variable:

$$\{\sin(x),\ \sin(2x),\ \sin(3x)\}.$$

$$\lambda_1 \sin(x) + \lambda_2 \sin(2x) + \lambda_3 \sin(3x) \equiv 0 \qquad (*)$$

Let $x = \pi/2 \Rightarrow \boxed{\lambda_1 - \lambda_3 = 0}$ — I

$\frac{d(*)}{dx} \Rightarrow \lambda_1 \cos(x) + 2\lambda_2 \cos(2x) + 3\lambda_3 \cos(3x) = 0$

Let $x = 0 \Rightarrow \boxed{\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0}$ — II

Let $x = \pi/2 \Rightarrow \boxed{2\lambda_2(-1) = 0}$ — III

III $\Rightarrow \boxed{\lambda_2 = 0}$

I $\Rightarrow \lambda_1 = \lambda_3$

$\leftarrow$ substitute into II $\Rightarrow$

$\lambda_1 + 2(0) + 3\lambda_1 = 0$

$\Rightarrow 4\lambda_1 = 0$

$\Rightarrow \boxed{\lambda_1 = 0} \Rightarrow \boxed{\lambda_3 = 0}$

So $(*) \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$

$\Rightarrow \{\sin(x), \sin(2x), \sin(3x)\}$ is linearly independent.

Q5]... [20 points] Let $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. Show that the collection H of all 2×2 matrices B such that $AB = 0$ is a subspace of $M_{2\times 2}$.

① $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in this collection since $A\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\Rightarrow$ nonempty

② if B_1, B_2 are in the collection $\Rightarrow AB_1 = 0 = AB_2$

$\Rightarrow A(B_1 + B_2) = AB_1 + AB_2 = 0 + 0 = 0$

$\Rightarrow B_1 + B_2$ in collection.

③ if B in collection & $\lambda \in \mathbb{R} \Rightarrow$

$A(\lambda B) = \lambda(AB) = \lambda 0 = 0$ $\Rightarrow \lambda B$ in collection

①,②,③ $\Rightarrow$ collection H is a subspace of $M_{2\times 2}$.

Find a linearly independent spanning set for H.

If $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ is in the collection H

then $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} b_1 - b_3 & b_2 - b_4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow b_1 = b_3$ & $b_2 = b_4$

$\Rightarrow B = \begin{pmatrix} b_1 & b_2 \\ b_1 & b_2 \end{pmatrix} = b_1 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

$\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ spans the collection H.

These are clearly lin indep. ... $\left[\lambda_1 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right.$

$\left. \Rightarrow \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \boxed{\lambda_1 = \lambda_2 = 0} \right]$