Sp'14 Sol'ns

Q1]... [15 points] Compute the following derivatives.
Find y' where

$$y = x^2 \cos(3x+1)$$

$$y' = 2x \cdot \cos(3x+1) + x^2(-\sin(3x+1)) \cdot 3 \qquad \text{(Product Rule)}$$

$$= 2x \cdot \cos(3x+1) - 3x^2 \sin(3x+1) \qquad \text{(Ch. Rule)}$$

Find the derivative y' where y is defined implicitly by the equation

$$y = \sin(2x+3y) \qquad \text{(Implicit Diff)}$$

$$y' = \cos(2x+3y) \cdot \frac{d}{dx}(2x+3y)$$

$$= \cos(2x+3y) \cdot (2+3y')$$

$$y'(1 - 3\cos(2x+3y)) = 2\cos(2x+3y)$$

$$y' = \frac{2\cos(2x+3y)}{1 - 3\cos(2x+3y)}$$

Suppose that $f(x)$ has derivatives of all orders. Find an expression for the nth derivative $g^{(n)}(x)$ where

$$g(x) = xf(x)$$

$$g'(x) = 1.f(x) + xf'(x)$$

$$g''(x) = 1.f'(x) + 1.f'(x) + xf''(x)$$

$$= 2f'(x) + xf^{(2)}(x)$$

$$g^{(3)}(x) = 2f^{(2)}(x) + 1.f^{(2)}(x) + xf^{(3)}(x)$$

$$= 3f^{(2)}(x) + xf^{(3)}(x)$$

Pattern:

$$g^{(n)}(x) = nf^{(n-1)}(x) + xf^{(n)}(x)$$

Q2]... [15 points] Find the following finite or infinite limits.

- $\lim\limits_{x\to\infty} \sqrt{x+1} - \sqrt{x}$

$$= \lim_{x\to\infty} \left(\frac{\sqrt{x+1} - \sqrt{x}}{1}\right)\left(\frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}\right)$$

$$= \lim_{x\to\infty} \frac{(x+1) - (x)}{\sqrt{x+1} + \sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} = 0$$

- $\lim\limits_{x\to 8} \dfrac{\sqrt[3]{x} - 2}{x - 8}$

$$= f'(8) \quad \text{where} \quad f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$$

$$\text{Ans} = \frac{1}{3} x^{-2/3}\Big|_{x=8} = \frac{1}{3(\sqrt[3]{8})^2} = \frac{1}{12}$$

- $\lim\limits_{\theta\to 0} \dfrac{\sin 3\theta}{2\theta + \tan 3\theta}$

$$= \lim_{\theta\to 0} \left(\frac{\sin(3\theta)}{3\theta} \frac{1}{\frac{2\theta}{3\theta} + \frac{\tan 3\theta}{3\theta}}\right)$$

$$= \lim_{\theta\to 0} \left(\frac{\sin(3\theta)}{3\theta} \cdot \frac{1}{\frac{2}{3} + \frac{\sin(3\theta)}{3\theta} \cdot \frac{1}{\cos(3\theta)}}\right)$$

$$= 1 \cdot \frac{1}{\frac{2}{3} + 1} = \frac{1}{\frac{5}{3}} = \frac{3}{5}$$

Q3]... [15 points] Evaluate the following integrals.

- The first is an indefinite integral.

$$\int 3x \sin(x^2+4)\,dx$$

Let $u = x^2+4 \Rightarrow \frac{du}{dx} = 2x \Rightarrow \frac{du}{2} = x\,dx$

$$\int = \int 3\sin(u)\,\frac{du}{2} = \frac{3}{2}(-\cos(u)) + C$$

$$= -\frac{3}{2}\cos(x^2+4) + C$$

- In the following definite integral $a > 0$ is a constant.

$$\int_0^a x\sqrt{a^2-x^2}\,dx$$

Let $u = a^2 - x^2$

When $\cancel{x} = 0 \Rightarrow u = a^2$

When $x = a \Rightarrow u = a^2 - a^2 = 0$

$\Rightarrow \frac{du}{dx} = -2x$

$\frac{du}{-2} = x\,dx$

$$\Rightarrow \int = \int_{a^2}^{0} u^{\frac{1}{2}}\,\frac{du}{-2}$$

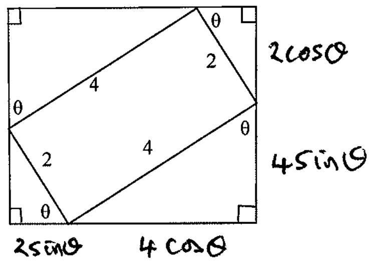
$$= \int_0^{a^2} u^{\frac{1}{2}}\,\frac{du}{2}$$

$$= \frac{1}{\cancel{2}}\frac{u^{3/2}}{3/\cancel{2}}\Bigg]_0^{a^2} = \frac{1}{3}\left((a^2)^{3/2} - 0\right)$$

$$= \frac{1}{3}a^3$$

Q4]... **[15 points]** Find the maximum area of a rectangle which can be circumscribed about a 2 by 4 rectangle. Follow the steps below.

- Find the length and breadth of the circumscribed rectangle as functions of θ. Note that $0 \le \theta \le \pi/2$.



length $= 2\cos\theta + 4\sin\theta$

width $= 2\sin\theta + 4\cos\theta$

(OR vice versa!)

length ⟷ width interchange is OK.

- Find the area $A(\theta)$ of a circumscribed rectangle as a function of θ.

$$A(\theta) = (2\cos\theta + 4\sin\theta)(2\sin\theta + 4\cos\theta)$$

$$= 4\cos\theta\sin\theta + 16\cos\theta\sin\theta + \underbrace{8\cos^2\theta + 8\sin^2\theta}_{=8}$$

$$= 20\cos\theta\sin\theta + 8$$

- Find the maximum area $A(\theta)$.

$$\frac{dA}{d\theta} = 20(-\sin\theta)(\sin\theta) + 20(\cos\theta)(\cos\theta) + \underbrace{0}_{\frac{d8}{d\theta}}$$

$$= 20(\cos^2\theta - \sin^2\theta)$$

$$\frac{dA}{d\theta} = 0 \Rightarrow \cos^2\theta - \sin^2\theta = 0$$

$$\Rightarrow \cos^2\theta = \sin^2\theta$$

$$\Rightarrow \cos\theta = \sin\theta \quad \text{... since } 0 \le \theta \le \pi/2$$

$$\Rightarrow \cos\theta = \sin\theta = \frac{1}{\sqrt{2}} \quad \text{.....}(\cos^2 + \sin^2 = 1)$$

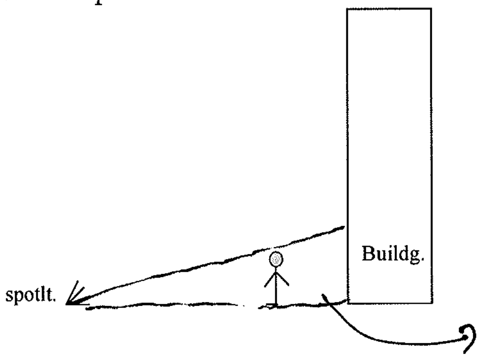
$$[\theta = \pi/4]$$

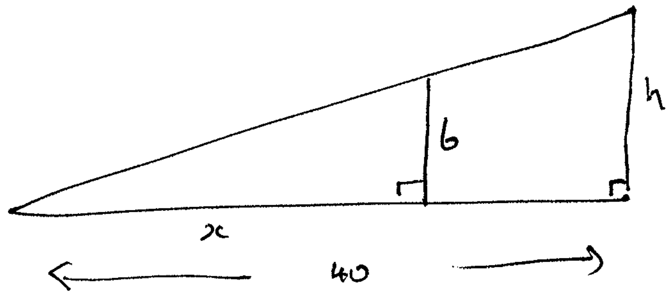
$$A_{Max} = A(\pi/4) = 20\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 8$$

$$= 10 + 8$$

$$= 18$$

Q5]... [15 points] A spotlight at ground level is located 40 feet from a very tall vertical building, directly in front of the door to the building. A 6 foot tall person leaves the building and walks directly towards the spotlight at 3 feet per second. How fast is the length of the person's shadow on the building changing when the person is 10 feet from the door of the building?





Let $\begin{cases} x = \text{distance from person to spotlight.} \\ h = \text{height of shadow on building.} \end{cases}$

Similar Δ's $\Rightarrow$ $\dfrac{h}{6} = \dfrac{40}{x}$ $\Rightarrow$ $hx = 6(40)$

$$\frac{d}{dt}(hx) = \frac{d}{dt}\big(6(40)\big) = 0$$

Product Rule

$$\Rightarrow \frac{dh}{dt}x + h\frac{dx}{dt} = 0$$

$$\Rightarrow \frac{dh}{dt} = -\frac{h}{x}\frac{dx}{dt}$$

We're told $\boxed{\dfrac{dx}{dt} = -3}$

note sign (x is decreasing)

When person is 10 ft from building, $x = 40 - 10 = 30$.

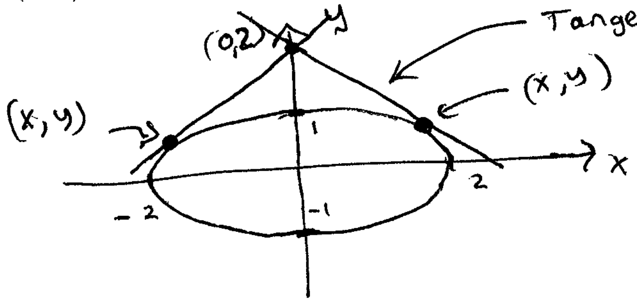
& $\dfrac{h}{6} = \dfrac{40}{30} = \dfrac{4}{3}$ $\Rightarrow$ $\boxed{h = \dfrac{(4)(6)}{3} = 8}$

$$\frac{dh}{dt} = -\frac{8}{30}(-3)$$

$$= \frac{8}{10}$$

$$= \frac{4}{5} \text{ feet/sec.}$$

Q6]... [15 points] Find the points on the ellipse $x^2 + 4y^2 = 4$ where the tangent lines contain the point $(0, 2)$.



Tangent Line contains the point $(0,2)$.

$\Downarrow$

$$\boxed{\text{Slope} = \frac{y-2}{x-0}} \quad \text{—①}$$

Tangent Line to ellipse at (x,y) has slope given by y'....

Implicit Diff. $\quad \frac{d}{dx}(x^2 + 4y^2) = \frac{d}{dx}(4)$

$$\Rightarrow 2x + 8y\,y' = 0$$

$$\Rightarrow y' = -\frac{2x}{8y} \Rightarrow \boxed{y' = \frac{-x}{4y}} \quad \text{—②}$$

Compare ① & ② for the slope of same tangent line!

$$\frac{y-2}{x} = \frac{-x}{4y}$$

$$4y^2 - 8y = -x^2$$

$$\Rightarrow x^2 + 4y^2 = 8y$$

use ellipse equation on LHS

$$\Rightarrow 4 = 8y$$

$$\Rightarrow y = \frac{4}{8} = \frac{1}{2} \Rightarrow x^2 + 4\left(\tfrac{1}{2}\right)^2 = 4$$

$$\Rightarrow x^2 + 1 = 4$$

$$\Rightarrow x = \pm\sqrt{3}$$

<u>Ans:</u> $(-\sqrt{3}, \tfrac{1}{2})$ and $(\sqrt{3}, \tfrac{1}{2})$

Makes sense from diagram above.

Q7]...[15 points] Compute the area of the region bounded by the curves $y = x^3$ and $y = \sqrt{x}$.

$y = x^3 \quad y = \sqrt{x}$

$\Downarrow$

$\sqrt{x} = x^3 \Rightarrow x = x^6$

$\Rightarrow x(1 - x^5) = 0$

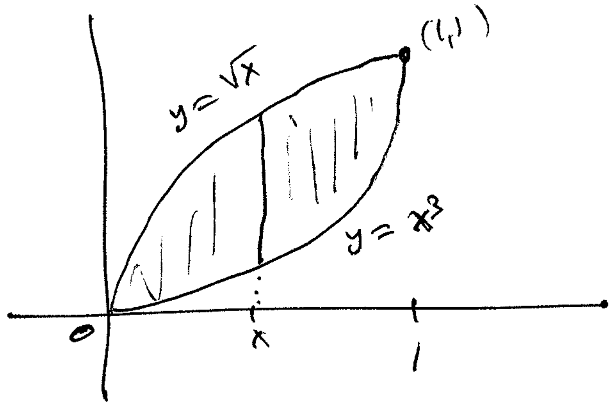
$x = 0$ or $x = 1$

$\Downarrow \quad \Downarrow$

$y = 0 \quad y = 1$

$(0,0)$ & $(1,1)$

intersect in ↗



$$\text{Area} = \int_0^1 (\text{Cross sectional length at } x)\, dx$$

$$= \int_0^1 (\sqrt{x} - x^3)\, dx$$

$$= \int_0^1 x^{\frac{1}{2}} - x^3\, dx$$

$$= \left[\frac{x^{3/2}}{3/2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{4} - 0$$

$$= \frac{8-3}{12} = \frac{5}{12}$$

Q8]... [15 points] Suppose that the derivative of a function f is given as $f'(x) = 3x^4 - 8x^3 + 6x^2$.

- Find the intervals where f is increasing and the intervals where f is decreasing.

Solve $\boxed{f'(x) = 0}$ $\quad x^2(3x^2 - 8x + 6) = 0 \rightarrow x = \dfrac{8 \pm \sqrt{64 - 4(3)(6)}}{12}$ ⊖

No Real Roots!

$\Rightarrow \boxed{x=0}$

$\Rightarrow 3x^2 - 8x + 6$ never crosses x-axis.

Note $3(0)^2 - 8(0) + 6 = 6 > 0$

$\Rightarrow 3x^2 - 8x + 6$ is <u>POSITIVE</u> for all x. ⊕

Since $(3x^2 - 8x + 6)$ ⊕

Test intervals

	$(-\infty, 0)$	$(0, \infty)$
$\text{sign}(f'(x)) = \text{sign}(x^2)$	+	+
$f(x)$	inc.	inc.

- Find the x–coordinates of the local minima of f, and find the x–coordinates of the local maxima of f.

By 1st part $f(x)$ is increasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ $\Rightarrow$ The only critical point $x=0$ is <u>Neither</u> a local max <u>nor</u> a local min.

<u>Ans</u>: None.

- Find the intervals where f is concave up, and the intervals where f is concave down.

$$f''(x) = \frac{d}{dx}(3x^4 - 8x^3 + 6x^2) = 12x^3 - 24x^2 + 12x$$
$$= 12x(x^2 - 2x + 1)$$
$$= 12x(x-1)^2$$

$f''(x) = 0 \Rightarrow$ <u>$x = 0, x = 1$</u>

Test Intervals

	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f''(x)$	$-$	+	+
$f(x)$	C.D.	C.U.	C.U

- Find the x–coordinates of the points of inflection of f.

By 3rd part there are two points $(x=0, x=1)$ where $f'' = 0$ but only one where f'' changes sign; namely, at 0.

<u>Ans</u>: $\boxed{x=0}$ is x-coordinate of point of inflection