Friday 02/14/2014 | Midterm I | 50 minutes

Name: [   ] Student ID: [   ]

Instructions.

1. Attempt all questions.
2. Do not write on back of exam sheets. Extra paper is available if you need it.
3. Show all the steps of your work clearly.
4. No calculators, no notes, no books.

Question	Points	Your Score
Q1	25	
Q2	25	
Q3	25	
Q4	25	
TOTAL	100	

Q1]... [25 points] The temperature T at a point (x, y) in a planar region is given by the function $T(x, y) = 4x^2 - y^2$. Suppose that units of measurement along the x- and y-axes are in inches.

(a) Draw some isotherms (curves of constant temperature), including the curves where $T = 0$.

(b) In what direction should an ant located at the point $(2, 1)$ move if it wishes to cool off as quickly as possible?

(c) If the ant starts walking at 2 inches per second in the direction you found above, what is the rate of change of temperature that the ant experiences? (Your answer will be in degrees per second.)

Q2]...[25 points] Suppose $z = f(x, y)$ has continuous second order partial derivatives, and suppose that $x = s^2 - t^2$ and $y = 2st$.

(a) Find an expression for z_s (that is, for $\frac{\partial z}{\partial s}$) in terms of the first order partial derivatives, z_x and z_y, of z with respect to x and y.

(b) Find an expression for z_t in terms of the first order partial derivatives of z with respect to x and y.

(c) Find an expression for z_{st} (that is, for $\frac{\partial^2 z}{\partial t \partial s}$) in terms of the first and second order partial derivatives of z with respect to x and y.

Q3]... [25 points] State the second derivative test for the function $f(x, y)$ at the critical point (a, b).

The function $f(x, y) = x^3 + y^3 - 3xy$ has two critical points. Find these critical points, and then use the second derivative test to classify them.

Q4]... [25 points] Find the equation of the tangent plane to the surface $xyz + 30 = 0$ at the point $(-3, 2, 5)$. Show your work.

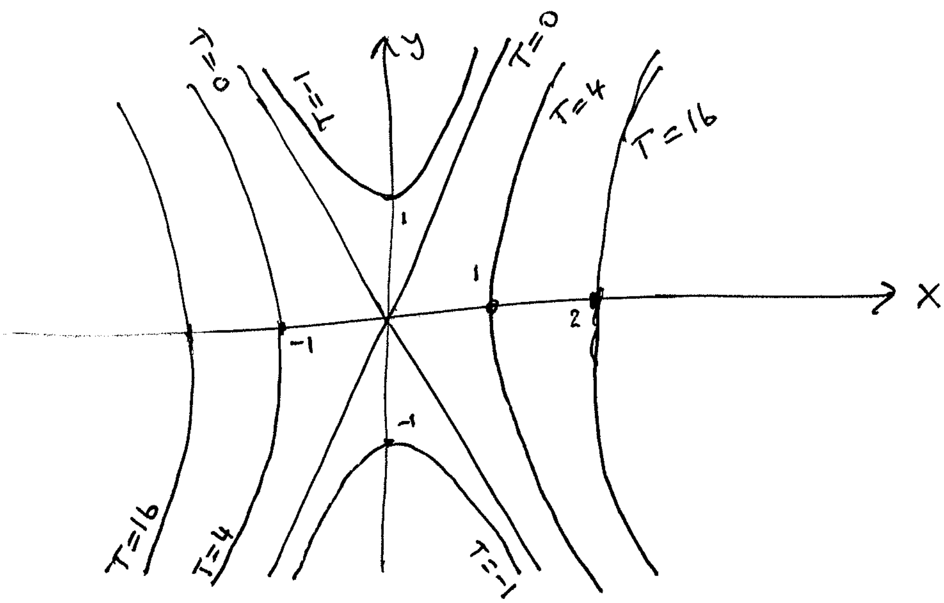
The point $(-3, 2, 5)$ also lies on the surface $z = x^2 - y^2$. The two surfaces $xyz + 30 = 0$ and $z = x^2 - y^2$ intersect in a curve passing through $(-3, 2, 5)$. Find a **tangent vector** to this curve at the point $(-3, 2, 5)$. (Hint: Such a tangent vector lies in the tangent plane to the surface $xyz + 30 = 0$ at the point $(-3, 2, 5)$; it **also** lies in the tangent plane to the surface $z = x^2 - y^2$ at the point $(-3, 2, 5)$.)

Q1]... [25 points] The temperature T at a point (x, y) in a planar region is given by the function $T(x, y) = 4x^2 - y^2$. Suppose that units of measurement along the x- and y-axes are in inches.

(a) Draw some isotherms (curves of constant temperature), including the curves where $T = 0$.

$T = 0$
$\Rightarrow (2x-y)(2x+y) = 0$
$\Rightarrow$ 2 Lines (shown)

$T = \text{const} \neq 0$
$\Rightarrow 4x^2 - y^2 = \text{const}$
(hyperbola)



(b) In what direction should an ant located at the point $(2, 1)$ move if it wishes to cool off as quickly as possible?

Want max decrease in T

$\Rightarrow -\nabla T_{(2,1)}$ direction

This is $-\nabla T_{(2,1)} = -\langle T_x, T_y \rangle_{(2,1)}$

$= -\langle 8x, -2y \rangle_{(2,1)} = -\langle 16, -2 \rangle$

$= \langle -16, 2 \rangle$

(c) If the ant starts walking at 2 inches per second in the direction you found above, what is the rate of change of temperature that the ant experiences? (Your answer will be in degrees per second.)

Rate of change of T (per unit distance traveled) $= -|\nabla T_{(2,1)}|$

$= -\sqrt{(16)^2 + 2^2}$ degrees/inch

Ant moves at constant speed 2 inches/sec.

$\Rightarrow$ Rate of change of T w.r.t. time $= 2\left(-\sqrt{(16)^2 + 4}\right)$

$= -4\sqrt{65}$ degrees/second.

Q2]... **[25 points]** Suppose $z = f(x,y)$ has continuous second order partial derivatives, and suppose that $x = s^2 - t^2$ and $y = 2st$.

(a) Find an expression for z_s (that is, for $\frac{\partial z}{\partial s}$) in terms of the first order partial derivatives, z_x and z_y, of z with respect to x and y.

Ch. Rule

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= z_x(2s) + z_y(2t)$$

$$\frac{\partial x}{\partial s} = 2s \qquad \frac{\partial y}{\partial s} = 2t \qquad \frac{\partial x}{\partial t} = -2t \qquad \frac{\partial y}{\partial t} = 2s$$

(b) Find an expression for z_t in terms of the first order partial derivatives of z with respect to x and y.

Ch. Rule

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= z_x(-2t) + z_y(2s)$$

(c) Find an expression for z_{st} (that is, for $\frac{\partial^2 z}{\partial t \partial s}$) in terms of the first and second order partial derivatives of z with respect to x and y.

$$\frac{\partial^2 z}{\partial t \partial s} = \frac{\partial}{\partial t}(z_s) = \frac{\partial}{\partial t}\left(z_x(2s) + z_y(2t)\right)$$

Sum + Product Rules →

$$= (2s)\frac{\partial z_x}{\partial t} + 2\frac{\partial t}{\partial t} z_y + (2t)\frac{\partial z_y}{\partial t}$$

Ch. Rule (Using the pattern obtained for $\frac{\partial z}{\partial t}$ in (b) above & replacing z with z_x & z_y) →

$$= (2s)\left((z_x)_x(-2t) + (z_x)_y(2s)\right)$$

$$+ 2z_y$$

$$+ (2t)\left((z_y)_x(-2t) + (z_y)_y(2s)\right)$$

$$= 4st\left(z_{yy} - z_{xx}\right) + 4(s^2 - t^2)z_{xy} + 2z_y$$

Q3]... [25 points] State the second derivative test for the function $f(x, y)$ at the critical point (a, b).

Given $f_x(a,b) = 0 = f_y(a,b)$

Let $D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$

(1) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then local MIN at (a,b).

(2) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then Local Max at (a,b).

(3) If $D(a,b) < 0$, then SADDLE at (a,b) (Neither local Max nor Min)

((4) $D(a,b) = 0 \Rightarrow$ No conclusion)

The function $f(x, y) = x^3 + y^3 - 3xy$ has two critical points. Find these critical points, and then use the second derivative test to classify them.

$f_x = 3x^2 - 3y \qquad f_y = 3y^2 - 3x$

$\left.\begin{array}{l} f_x = 0 \\ f_y = 0 \end{array}\right) \Rightarrow \left.\begin{array}{l} x^2 = y \\ y^2 = x \end{array}\right) \Rightarrow x^4 - x = 0$

$x(x^3 - 1) = 0 \Rightarrow x = 0,\ x = 1$

$x = 0 \Rightarrow y = 0^2 = 0 \qquad x = 1 \Rightarrow y = 1^2 = 1$

C.P.'s are $(0,0)$ and $(1,1)$

$$D = f_{xx} f_{yy} - (f_{xy})^2$$
$$= (6x)(6y) - (-3)^2 = 36xy - 9$$

(1) $D(0,0) = -9 < 0 \Rightarrow$ SADDLE at $(0,0)$

(2) $D(1,1) = 36 - 9 > 0$ & $f_{xx}(1,1) = 6(1) > 0 \Rightarrow$ LOCAL MIN at $(1,1)$

Q4]...[**25 points**] Find the equation of the tangent plane to the surface $xyz + 30 = 0$ at the point $(-3, 2, 5)$. Show your work.

Let $F(x,y,z) = xyz + 30$. Our surface is the level surface $F = 0$.

$\nabla F_{(-3,2,5)}$ gives normal vector for t. plane.

$$\nabla F_{(-3,2,5)} = \langle F_x, F_y, F_z \rangle_{(-3,2,5)} = \langle yz, xz, xy \rangle_{(-3,2,5)} = \langle 10, -15, -6 \rangle$$

So Equation of tangent plane is

$$\langle 10, -15, -6 \rangle \cdot \langle x-(-3), y-2, z-5 \rangle = 0$$

OR

$$\boxed{10(x+3) - 15(y-2) - 6(z-5) = 0}$$

The point $(-3, 2, 5)$ also lies on the surface $z = x^2 - y^2$. The two surfaces $xyz + 30 = 0$ and $z = x^2 - y^2$ intersect in a curve passing through $(-3, 2, 5)$. Find a **tangent vector** to this curve at the point $(-3, 2, 5)$. (Hint: Such a tangent vector lies in the tangent plane to the surface $xyz + 30 = 0$ at the point $(-3, 2, 5)$; it **also** lies in the tangent plane to the surface $z = x^2 - y^2$ at the point $(-3, 2, 5)$.)

Tangent vector lies in both tangent planes $\Rightarrow$ is $\perp$ to both normals

So cross product of normals will give us a tangent vector.

1^{st} normal $= \langle 10, -15, -6 \rangle$.

2^{nd} surface is $G = 0$, where $G(x,y,z) = x^2 - y^2 - z$.

$$2^{nd} \text{ normal} = \nabla G_{(-3,2,5)} = \langle G_x, G_y, G_z \rangle_{(-3,2,5)}$$

$$= \langle 2x, -2y, -1 \rangle_{(-3,2,5)}$$

$$= \langle -6, -4, -1 \rangle$$

Tangent vector $= \langle 10, -15, -6 \rangle \times \langle -6, -4, -1 \rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10 & -15 & -6 \\ -6 & -4 & -1 \end{vmatrix} = \langle 15-24, 36+10, -40-90 \rangle$$

$$= \langle -9, 46, -130 \rangle$$

OR any nonzero multiple