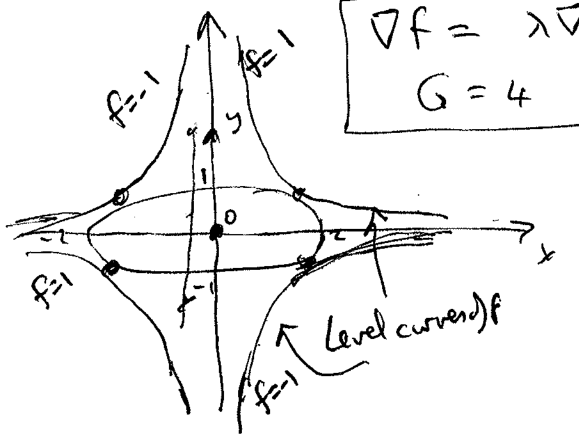
SP'14 — CALC IV — FINAL

Q1]... [15 points] Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = xy$ on the ellipse $\underbrace{x^2 + 4y^2}_{G(x,y)} = 4$.



$$\boxed{\begin{matrix}\nabla f = \lambda \nabla G \\ G = 4\end{matrix}} \Rightarrow \left[\begin{matrix} y = \lambda(2x) \\ x = \lambda(8y) \\ x^2+4y^2 = 4\end{matrix}\right] \Rightarrow y = \lambda^2(16)y$$

$$\Downarrow$$

$$\boxed{y(1-16\lambda^2) = 0}$$

$\underline{y=0} \Rightarrow x = 8\lambda(0) = 0$ NOT A SOLUTION [(0,0) doesn't make G = 4.]

OR

$1 - 16\lambda^2 = 0$ $\quad \lambda = \pm\frac{1}{4}$

$\boxed{\lambda = \frac{1}{4}}$

$x = 2y$

$\Rightarrow (2y)^2 + 4y^2 = 4$

$8y^2 = 4$

$y^2 = \frac{1}{2}$

$y = \pm\frac{1}{\sqrt{2}}$ $\quad x = 2y = \pm\sqrt{2}$ $\quad \boxed{\pm\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right)}$

$\lambda = -\frac{1}{4}$ $\quad x = -2y$

$(-2y)^2 + 4y^2 = 4$

$y = \pm\frac{1}{\sqrt{2}}$

$x = -2y = \mp\sqrt{2}$ $\quad \boxed{\pm\left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right)}$

At $\pm\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right) \ldots f(x,y) = \boxed{1}$ ← Max val.

At $\pm\left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right) \ldots f(x,y) = \boxed{-1}$ ← Min val.

Q2]... [15 points] This question asks about how the function $f(x, y, x) = x^2 - xy + yz$ is changing at the point $(1, 2, 1)$.

- Find the direction in which f is increasing the fastest at the point $(1, 2, 1)$.

Ans $\nabla f_{(1,2,1)} = \langle f_x(1,2,1), f_y(1,2,1), f_z(1,2,1) \rangle$

$= \langle 2x - y, -x + z, y \rangle_{(1,2,1)}$

$= \underline{\langle 0, 0, 2 \rangle}$

- What is this maximum rate of increase of f at $(1, 2, 1)$?

Ans $= |\nabla f_{(1,2,1)}| = |\langle 0, 0, 2 \rangle| = \boxed{2}$

- Find all the directions in which f is not changing at $(1, 2, 1)$.

Such directions $\hat{u} = \langle a, b, c \rangle$ satisfy

$D_{\hat{u}} f(1,2,1) = 0$

ie. $\nabla f_{(1,2,1)} \cdot \hat{u} = 0$ $\qquad \langle a, b, c \rangle \cdot \langle 0, 0, 2 \rangle = 0$

$c = 0$

$\boxed{\text{ANY} \to \langle a, b, 0 \rangle \text{ where } a^2 + b^2 = 1}$.

- Find all the directions in which the rate of change of f at $(1, 2, 1)$ is 50% of the maximum rate in part 2 above.

Find $\hat{u} = \langle a, b, c \rangle$ so that

$|D_{\hat{u}} f(1,2,1)| = |\nabla f_{(1,2,1)} \cdot \hat{u}| = \frac{2}{2}$

$\langle 0, 0, 2 \rangle \cdot \langle a, b, c \rangle = 1$ $\qquad 2c = 1$ $\qquad c = \frac{1}{2}$

& $a^2 + b^2 + c^2 = 1$

$\Rightarrow a^2 + b^2 = 1 - \frac{1}{4} = \frac{3}{4}$

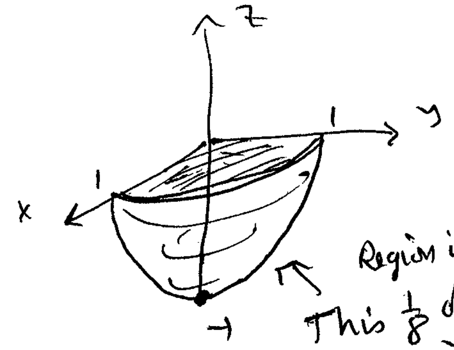
Ans $\boxed{\text{all } \langle a, b, \tfrac{1}{2} \rangle \text{ where } a^2 + b^2 = 3/4}$ $\leftarrow$ Cone of angle $\pi/3$ about $\nabla f_{(1,2,1)}$ direction!

Q3]... [15 points] Write down **triple integral** expressions for the volume integral of a function $f(x, y, x)$ over the region which is inside the unit sphere $x^2 + y^2 + z^2 = 1$ and also in the octant where $x \geq 0$, $y \geq 0$ and $z \leq 0$ (note that the region is one eighth of a solid unit ball, and it lies below the xy–plane). Give three versions of your answer, one for each of the following coordinate systems.

Cylindrical Coordinates.

$$\int_0^{\frac{\pi}{2}} \int_0^1 \int_{-\sqrt{1-x^2-y^2}}^{0} f(r\cos\theta, r\sin\theta, z)\, r \cdot dz\, dr\, d\theta$$

$-\sqrt{1-r^2}$



Region is This $\frac{1}{8}$ of ball.

$$z^2 = 1 - x^2 - y^2$$
$$z = -\sqrt{1-x^2-y^2} \quad (z \leq 0)$$

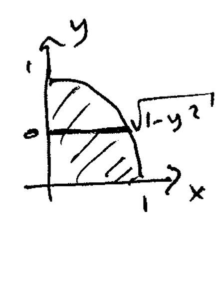
Spherical Coordinates.

$$\int_0^{\pi/2} \int_{\pi/2}^{\pi} \int_0^1 f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Note: ϕ Limits are $\pi/2$ to π — "Lower" portion of ball.

Cartesian Coordinates.

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{-\sqrt{1-x^2-y^2}}^{0} f(x,y,z)\, dz\, dx\, dy$$

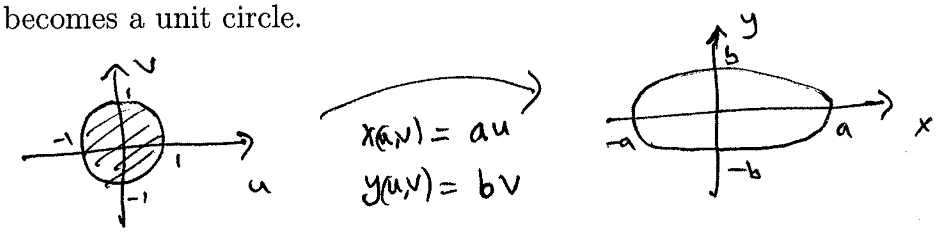


Q4]... [15 points] Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Here $a > 0$ and $b > 0$ are constants.

- Find a suitable change of coordinates $x = x(u,v)$, $y = y(u,v)$ which converts the ellipse into a unit circle. Write down your change of coordinate functions $x = x(u,v)$, $y = y(u,v)$ explicitly, and verify that the ellipse becomes a unit circle.



$$\left.\begin{array}{l} x = au \\ y = bv \end{array}\right) \Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{becomes} \quad \frac{\cancel{a^2} u^2}{\cancel{a^2}} + \frac{\cancel{b^2} v^2}{\cancel{b^2}} = 1 \Rightarrow \boxed{u^2 + v^2 = 1}$$

unit circle.

- Use the change of coordinates formula for multiple integrals to compute the area inside the ellipse above.

$$\iint_{\text{ellipse}} 1 \, dx\,dy = \iint_{\substack{\text{unit}\\ \text{disk}}} 1 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \iint_{\substack{\text{unit}\\ \text{disk}}} ab \, du\,dv = ab \left(\iint_{\substack{\text{unit}\\ \text{disk}}} du\,dv \right)$$

$$= ab\,(\pi 1^2)$$

$$= \boxed{\pi ab}$$

Computation of $\frac{\partial(x,y)}{\partial(u,v)}$ ...

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$= ab$$

Q5]... **[15 points]** Test to see if either of the vector fields **F** and **G** below is conservative. If a vector field is not conservative, say why not. If a vector field is conservative, express it as the gradient of some scalar field.

$$\mathbf{F} = \langle y^2,\, -z^2,\, x^2 \rangle$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -z^2 & x^2 \end{vmatrix} = \left\langle \frac{\partial x^2}{\partial y} - \frac{\partial(-z^2)}{\partial z},\ \text{OTHER STUFF} \right\rangle$$

$$= \langle 2z,\ \text{OTHER STUFF} \rangle$$

$$\neq \vec{0}$$

$\text{Curl}(\vec{F}) \neq \vec{0} \Rightarrow \vec{F}$ not conservative

$$\mathbf{G} = \langle x^3 - 3xy^2,\, y^3 - 3x^2y,\, z \rangle$$

$$\nabla \times \vec{G} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3 - 3xy^2 & y^3 - 3x^2y & z \end{vmatrix} = \langle 0-0,\ 0-0,\ -6xy-(-6xy) \rangle$$

$$= \langle 0, 0, 0 \rangle$$

$\text{Curl}(\vec{G}) = \vec{0}$ & $\vec{G}$ defined on all of $\mathbb{R}^3$ (NO HOLES) $\Rightarrow$

$G = \nabla f$ for some f (i.e. $\vec{G}$ is conservative)

$$\left.\begin{aligned} f_x &= x^3 - 3xy^2 \\ f_y &= y^3 - 3x^2y \\ f_z &= z \end{aligned}\right) \Rightarrow \boxed{f = \frac{x^4}{4} + \frac{y^4}{4} - \frac{3x^2y^2}{2} + \frac{z^2}{2} + C}$$

Compare all 3 antiderivatives etc....

constant.

Q6]...[15 points] Compute the flux integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

of the vector field

$$\mathbf{F} = \langle y, -x, z \rangle$$

where S is the surface described by

$$\mathbf{r}(u, v) = \langle u\cos(v), u\sin(v), v \rangle \qquad 0 \le u \le 2 \qquad 0 \le v \le 2\pi$$

oriented so as to have upward pointing normal.

$$\begin{array}{cccc} & \hat{i} & \hat{j} & \hat{k} \\ \vec{r}_u = & \langle \cos(v), & \sin(v), & 0 \rangle \\ \vec{r}_v = & \langle -u\sin(v), & u\cos(v), & 1 \rangle \end{array}$$

$$d\vec{S} = (\vec{r}_u \times \vec{r}_v)\,du\,dv = \langle \sin(v), -\cos(v), u(\cos^2(v)+\sin^2(v)) \rangle\, du\,dv$$

$$= \langle \sin(v), -\cos(v), u \rangle\, du\,dv$$

3rd component ≥ 0 $\Rightarrow$ "upward" pointing!

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S}$$

$$= \int_0^{2\pi}\int_0^2 \langle u\sin(v), -u\cos(v), v \rangle \cdot \langle \sin(v), -\cos(v), u \rangle\, du\,dv$$

$$= \int_0^{2\pi}\int_0^2 \left(u(\sin^2(v)+\cos^2(v)) + uv\right) du\,dv$$

$$= \int_0^{2\pi}\int_0^2 u(1+v)\, du\,dv$$

$$= \int_0^2 u\,du \int_0^{2\pi}(1+v)\,dv = \left[\frac{u^2}{2}\right]_0^2 \left[v + \frac{v^2}{2}\right]_0^{2\pi}$$

$$= 2(2\pi)\left(1 + \frac{2\pi}{2}\right) = \boxed{4\pi(1+\pi)}$$

Q7]... [15 points] Let $\mathbf{F}$ be the vector field $\mathbf{F} = \langle z, xyz, -x \rangle$.

- Compute Curl($\mathbf{F}$).

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & xyz & -x \end{vmatrix}$$

$$= \left\langle -xy, \ \frac{\partial z}{\partial z} - \frac{\partial(-x)}{\partial x}, \ yz \right\rangle$$

$$= \langle -xy, \ 2, \ yz \rangle$$

- Use Stokes' Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the circle $x^2 + z^2 = 9$, $y = 2$ which is the oriented boundary of the disk $x^2 + z^2 \leq 9$, $y = 2$ which is oriented with normal vector $\langle 0, 1, 0 \rangle$.

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{\partial D} \vec{F} \cdot d\vec{r}$$

Stokes'

$$= \iint_D \operatorname{Curl}(F) \cdot d\vec{S}$$

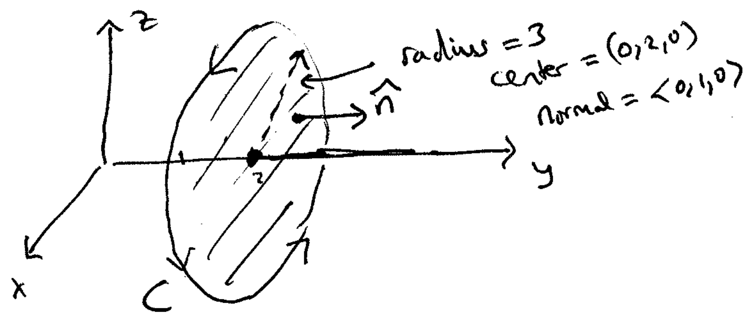
$$= \iint_D \langle xy, 2, yz \rangle \cdot \langle 0,1,0 \rangle \, dx\, dz$$

$$= \iint_D 2 \, dx\, dz$$

$$= 2 \iint_D dx\, dz$$

= 2 Area Disk of radius 3

$$= 2\left(\pi (3)^2\right) = \boxed{18\pi}$$



Let D = disk shaded, oriented to have normal $\hat{n} = \langle 0,1,0 \rangle$

$\partial D = C$ has given orientation

$$d\vec{S} = \langle 0,1,0 \rangle \, dx\, dz$$

vertical plane ($y = 2$)

Trivial parameterization!

$$\vec{r}(x,z) = \langle x, 2, z \rangle$$

$$\vec{r}_x = \langle 1,0,0 \rangle = \hat{\imath}, \quad \vec{r}_z = \langle 0,0,1 \rangle = \hat{k}$$

$d\vec{S} = (\vec{r}_z \times \vec{r}_x)\, dx\, dz$

change order

Q8]... **[15 points]** Use the Divergence Theorem to compute the flux integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

of the vector field

$$\mathbf{F} = \langle x^2 + \sin(y^2 z),\ y^2 - \cos(x+z),\ z^2 + e^x \rangle$$

where S is the boundary of the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ with **outward** pointing normal.

By Divergence Theorem ...

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\partial(\text{Cube})} \vec{F} \cdot ds = \iiint_{\text{Cube}} \operatorname{div}(\vec{F})\, dV \quad \text{—— } (*)$$

Now

$$\begin{aligned} \operatorname{div}(\vec{F}) &= P_x + Q_y + R_z \\ &= (2x+0) + (2y-0) + (2z+0) \\ &= 2x + 2y + 2z \end{aligned}$$

$$\begin{aligned} (*) &= \iiint_{[0,1]^3} 2x+2y+2z\, dV \\ &= 3 \iiint_{[0,1]^3} 2x\, dV \\ &= 3 \int_0^1\int_0^1\int_0^1 2x\, dx\, dy\, dz \\ &= 3 \int_0^1 \int_0^1 [x^2]_0^1\, dy\, dz \\ &= 3 \int_0^1 \int_0^1 1\, dy\, dz \\ &= 3\, [y]_0^1\, [z]_0^1 = \boxed{3} \end{aligned}$$

Now $x \to y \to z \to x$

is a symmetry of the unit cube (Rotation about diagonal from (0,0,0) to (1,1,1))

$$\Rightarrow \iiint_{\text{Cube}} 2x\, dV = \iiint_{\text{Cube}} 2y\, dV = \iiint_{\text{Cube}} 2z\, dV$$