Math 2443-002 Path Integrals Spring 2000

Path Integrals: Let $\mathbf{F}$ be a vector field defined on an open region containing a curve C given by $\mathbf{r}(t)$, $a \le t \le b$. In 2-dimensions we have $\mathbf{F} = \langle P, Q \rangle$ and $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. In 3-dimensions we have $\mathbf{F} = \langle P, Q, R \rangle$ and $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Here is the definition of path integral of $\mathbf{F}$ along C.

$$\begin{aligned}
\int_C \mathbf{F} \cdot \hat{\mathbf{T}} ds &= \int_C \mathbf{F} \cdot d\mathbf{r} \\
&= \begin{cases} \int_C P dx + Q dy & \text{(in 2-d)} \\ \int_C P dx + Q dy + R dz & \text{(in 3-d)} \end{cases} \\
&= \begin{cases} \int_a^b [P(x(t), y(t)) \frac{dx}{dt} + Q(x(t), y(t)) \frac{dy}{dt}] dt & \text{(in 2-d)} \\ \int_a^b [P(x(t), y(t), z(t)) \frac{dx}{dt} + Q(x(t), y(t), z(t)) \frac{dy}{dt} + R(x(t), y(t), z(t)) \frac{dz}{dt}] dt & \text{(in 3-d)} \end{cases}
\end{aligned}$$

Computing Path Integrals: There are three steps to computing the path integral of a vector field over a path C. The *hard steps* are actually from Calculus III and Calculus I/II.

- Find a parametric description of C [Calc III].
- Use the last part of the definition/equation above.
- Evaluate the resulting 1-variable integral [Calc I/II].

Fundamental theorem: Let f be a function of 2 or 3 variables, and let C: $\mathbf{r}(t)$, $a \le t \le b$ be a path which lies in the domain of f. Then we have

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Green's Theorem: Let C be a piecewise-smooth, positively oriented, simple closed curve in the plane which bounds a region D. If P and Q have continuous partial derivatives on an open region which contains D then

$$\int_C P dx + Q dy = \int\int_D [Q_x - P_y] \, dA.$$

Application of Green to areas: Let C and D be as in Green's theorem above. Then the area of the region D can be computed as follows

$$Area(D) = \int\int_D dA = \begin{cases} \int_C x dy & \text{or} \\ -\int_C y dx & \text{or} \\ \frac{1}{2} \int_C x dy - y dx \end{cases}$$