Q53
pge 956

SHOW $$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}}$$

Method I: (Work from RHS to LHS.)

FORMULA II

$x = r\cos\theta$

$\Rightarrow \frac{\partial x}{\partial r} = \cos\theta$

& $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$y = r\sin\theta$

$\Rightarrow \frac{\partial y}{\partial r} = \sin\theta$

& $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \qquad \text{— (1)}$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right)$$

$$= \cos\theta \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + \sin\theta \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$$

---using (1), sum Rule, & fact that $\theta = \text{const}$ when taking $\frac{\partial}{\partial r}$

$$= \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial r}\right) + \sin\theta\left(\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right)$$

$$= \cos\theta\left(z_{xx}\cos\theta + z_{xy}\sin\theta\right) + \sin\theta\left(z_{yx}\cos\theta + z_{yy}\sin\theta\right)$$

$$= z_{xx}\cos^2\theta + z_{yy}\sin^2\theta + 2z_{xy}\cos\theta\sin\theta \qquad \text{— (2)}$$

---using $z_{xy} = z_{yx}$ Result

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= -\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta \quad \text{—— (3)}$$

$$\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial \theta}\right)$$

$$= -\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right) r\sin\theta - \frac{\partial z}{\partial x} r\cos\theta$$

$$+ \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right) r\cos\theta - \frac{\partial z}{\partial y} r\sin\theta$$

... use (3), sum + product Rules & fact that $r =$ const. when taking $\frac{\partial}{\partial \theta}$

$$= -r\sin\theta\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial \theta}\right) - \frac{\partial z}{\partial x} r\cos\theta$$

$$+ r\cos\theta\left(\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial \theta}\right) - \frac{\partial z}{\partial y} r\sin\theta$$

$$= r^2\sin^2\theta\,\frac{\partial^2 z}{\partial x^2} \cancel{+} - r^2\cos\theta\sin\theta\, z_{xy} - z_x r\cos\theta$$

$$- r^2\cos\theta\sin\theta\, z_{xy} + r^2\cos^2\theta\, z_{yy} - z_y r\sin\theta$$

—— (4)

Substitute ①, ②, ④ into RHS to get

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}$$

$$= Z_{xx}\cos^2\theta + Z_{yy}\sin^2\theta + 2Z_{xy}\cos\theta\sin\theta + \frac{1}{r}\left(Z_x\cos\theta + Z_y\sin\theta\right)$$

$$+ \frac{1}{r^2}\left(Z_{xx}\, r^2\sin^2\theta + Z_{yy}\, r^2\cos^2\theta - 2Z_{xy}\, r^2\cos\theta\sin\theta \; \cancel{+}{-}\; r\left(Z_x\cos\theta + Z_y\sin\theta\right)\right)$$

cancel   cancel   cancel

$$= Z_{xx}\left(\cos^2\theta + \sin^2\theta\right) + Z_{yy}\left(\cos^2\theta + \sin^2\theta\right) + 0 + 0 + 0$$

$$= Z_{xx} + Z_{yy}$$

$$= \text{LHS}$$

done!

Method II (Work from LHS to RHS)

We will use expressions like

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} \quad \text{etc.} \dots$$

---. **So** we need to find r_x, r_y, θ_x, θ_y ----

This involves (at least implicitly) solving for r, θ in terms of x, y ....] →

$x = r\cos\theta \qquad y = r\sin\theta$

$$\Downarrow$$

$$x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta)$$

$$\Downarrow$$

$$\boxed{r = \sqrt{x^2+y^2}} \text{ — } \boxed{\text{I}}$$

$$\Rightarrow \boxed{\tan\theta = \frac{y}{x}} \text{ — } \boxed{\text{II}}$$

First → $\boxed{\frac{\partial}{\partial x} \ \& \ \frac{\partial}{\partial y} \text{ of } \boxed{\text{I}}}$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y = \frac{y}{r} = \sin\theta$$

Now → $\boxed{\frac{\partial}{\partial x} \ \& \ \frac{\partial}{\partial y} \text{ of } \boxed{\text{II}}}$

$\frac{\partial}{\partial x}$:

$$\sec^2\theta \frac{\partial\theta}{\partial x} = -\frac{y}{x^2}$$

$$\Rightarrow \frac{\partial\theta}{\partial x} = -\frac{y}{x^2}\cos^2\theta = -\frac{r\sin\theta}{r^2\cos^2\theta}\cos^2\theta = -\frac{\sin\theta}{r}$$

$\frac{\partial}{\partial y}$:

$$\sec^2\theta \frac{\partial\theta}{\partial y} = \frac{1}{x}$$

$$\Rightarrow \frac{\partial\theta}{\partial y} = \frac{1}{x}\cos^2\theta = \frac{\cos^2\theta}{r\cos\theta} = \frac{\cos\theta}{r}$$

Record These

$$\boxed{\begin{array}{ll} \frac{\partial r}{\partial x} = \cos\theta & \frac{\partial r}{\partial y} = \sin\theta \\ \frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r} & \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r} \end{array}} \text{ — } (*)$$

FORMULA II

Now we are ready to work from LHS to RHS.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial z}{\partial r}\cos\theta + \frac{\partial z}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial r}\cos\theta + \frac{\partial z}{\partial \theta}\left(\frac{-\sin\theta}{r}\right)\right)$$

Sum + product Rules

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial r}\right)\cos\theta + \frac{\partial z}{\partial r}\frac{\partial}{\partial x}(\cos\theta) - \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \theta}\right)\frac{\sin\theta}{r} - \frac{\partial z}{\partial \theta}\frac{\partial}{\partial x}\left(\frac{\sin\theta}{r}\right)$$

Chain Rule (multiple times)

$$= \left(\frac{\partial^2 z}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 z}{\partial\theta\partial r}\frac{\partial\theta}{\partial x}\right)\cos\theta + \frac{\partial z}{\partial r}\left(\cancelto{0}{\frac{\partial(\cos\theta)}{\partial r}\frac{\partial r}{\partial x}} + \frac{\partial(\cos\theta)}{\partial\theta}\frac{\partial\theta}{\partial x}\right)$$

$$- \left(\frac{\partial^2 z}{\partial r\partial\theta}\frac{\partial r}{\partial x} + \frac{\partial^2 z}{\partial\theta^2}\frac{\partial\theta}{\partial x}\right)\frac{\sin\theta}{r} - \frac{\partial z}{\partial\theta}\left(\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\right)\frac{\partial\theta}{\partial x}\right)$$

$$= z_{rr}\cos^2\theta - \frac{2\cos\theta\sin\theta}{r}z_{r\theta} + z_{\theta\theta}\frac{\sin^2\theta}{r^2}$$

$$+ z_r\frac{\sin^2\theta}{r} + 2z_\theta\frac{\cos\theta\sin\theta}{r^2} \quad \text{— (A)}$$

Similarly,

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial \theta}\frac{\cos\theta}{r}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial \theta}\frac{\cos\theta}{r}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial r}\right)\sin\theta + \frac{\partial z}{\partial r}\frac{\partial}{\partial y}(\sin\theta) + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \theta}\right)\frac{\cos\theta}{r} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial y}\left(\frac{\cos\theta}{r}\right)$$

$$= \left(\frac{\partial^2 z}{\partial r^2}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial \theta \partial r}\frac{\partial \theta}{\partial y}\right)\sin\theta + \frac{\partial z}{\partial r}\left(\cancelto{0}{\frac{\partial \sin\theta}{\partial r}}\frac{\partial r}{\partial y} + \frac{\partial \sin\theta}{\partial \theta}\frac{\partial \theta}{\partial y}\right)$$

$$+ \left(\frac{\partial^2 z}{\partial r \partial \theta}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial \theta^2}\frac{\partial \theta}{\partial y}\right)\frac{\cos\theta}{r} + \frac{\partial z}{\partial \theta}\left(\frac{\partial\left(\frac{\cos\theta}{r}\right)}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial\left(\frac{\cos\theta}{r}\right)}{\partial \theta}\frac{\partial \theta}{\partial y}\right)$$

$$= z_{rr}\sin^2\theta + 2\frac{\cos\theta\sin\theta}{r}z_{r\theta} + z_{\theta\theta}\frac{\cos^2\theta}{r^2}$$

$$+ z_r\frac{\cos^2\theta}{r} + z_\theta\left(-2\frac{\cos\theta\sin\theta}{r^2}\right) \quad \text{--- } B$$

Now

$$z_{xx} + z_{yy} \overset{A\,\&\,B}{=} z_{rr}(\cos^2\theta + \sin^2\theta) + z_{r\theta}(0)$$

$$+ z_{\theta\theta}\left(\frac{\cos^2\theta + \sin^2\theta}{r^2}\right) + z_r\left(\frac{\cos^2\theta + \sin^2\theta}{r}\right)$$

$$+ z_\theta(0)$$

$$= z_{rr} + \frac{1}{r}z_r + \frac{1}{r^2}z_{\theta\theta} = \text{RHS}$$

done!

Remark ①: " $\longrightarrow$ " vs. " $\longleftarrow$ "

You should compare the two methods ...

$x = r\cos\theta$, $y = r\sin\theta$ are explicitly given to us, and so $x_r, x_\theta, y_r, y_\theta$ are easy to compute.

Therefore $z_r = z_x x_r + z_y y_r$ etc are easy to compute.

This suggests working from RHS to LHS.
(Method I)

Note: Working from LHS to RHS took more work:

① We need to find (and to justify!) expressions for $r_x, r_y, \theta_x, \theta_y$.

② The calculations $z_x = z_r r_x + z_\theta \theta_x$ + resulting computations for 2nd derivatives z_{xx} etc are at least as hard as those in Method I

Morals (i) Resist the temptation to always work from left to Right!

(ii) Justify new expressions that magically appear in your solutions (eg r_x, θ_x etc..)

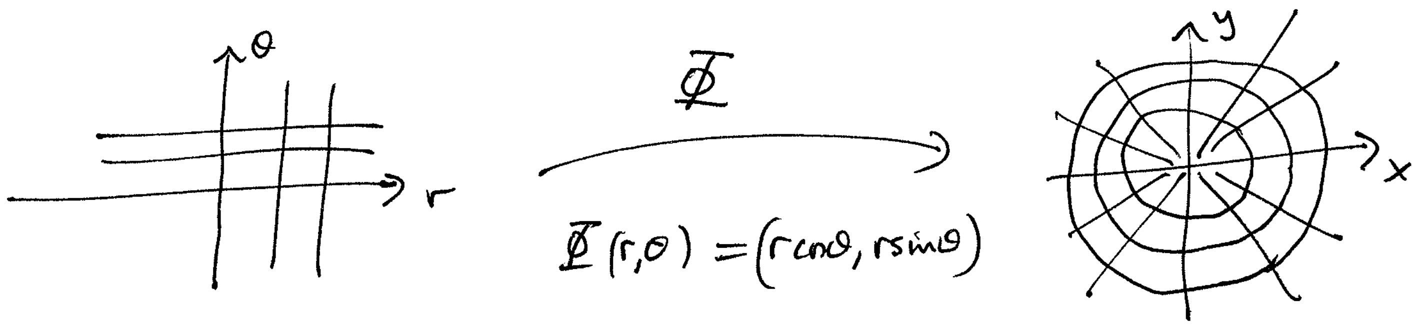
Remark (2): You might be surprised by the fact that

$$\frac{\partial r}{\partial x} = \cos\theta \neq \frac{1}{\frac{\partial x}{\partial r}} \; ; \; \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \neq \frac{1}{\frac{\partial x}{\partial \theta}}$$

etc...

Partial derivatives do not behave like ordinary (1-variable) derivatives in this respect $\left(\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} \right)$.

However, there <u>is</u> an analogue of $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$ in the multivariable world; but it involves matrices of partial derivatives.



The <u>DERIVATIVE</u>, $D\Phi$, is a <u>LINEAR FUNCTION</u> from $\mathbb{R}^2$ to $\mathbb{R}^2$. It is represented by a (2×2)-matrix of partial derivatives:

$$D\Phi = \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

From FORMULA I (PAGE 1)

$$D(\Phi^{-1}) = (D\Phi)^{-1} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}^{-1}$$

This is the analogue of $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$ for multivariable functions (suitable ones).

Now, from your linear algebra class you see that the inverse of a (2×2) matrix is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

In our case $\quad ad-bc = r\cos^2\theta - (-r\sin^2\theta)$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

& the inverse becomes ...

$$D\Phi^{-1} = \frac{1}{r}\begin{pmatrix} r\cos\theta & -(r\sin\theta) \\ -(\sin\theta) & \cos\theta \end{pmatrix}$$

$\Phi^{-1}(x,y) = (r(x,y), \theta(x,y))$

$D(\Phi^{-1}) = \begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix}$

$$\boxed{D\Phi^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}} \leftarrow \text{But this is } \begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix}$$

$\Rightarrow$ Reclaim Formula II !