Calculus III [2433–001] Homework 10.2: #57

This exercise gives a pretty geometric picture for the telescoping series

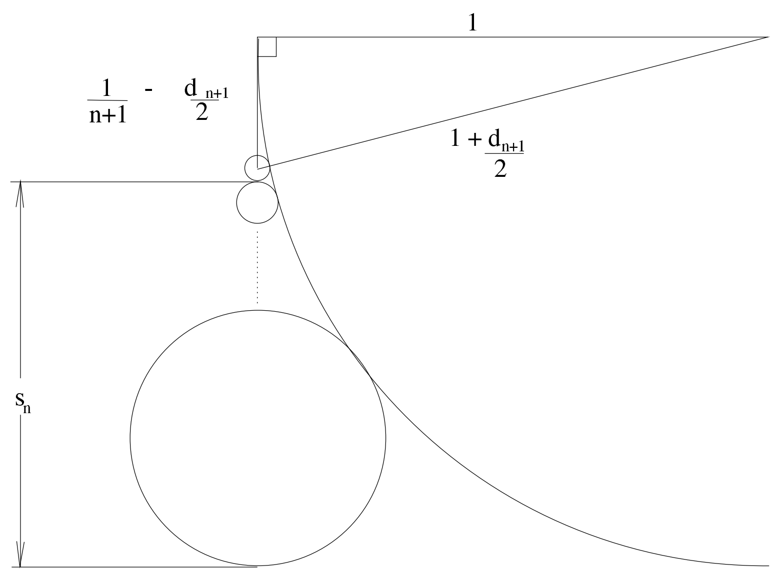
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Let d_n denote the diameters of the circles (counting from the bottom upwards) which are packed between the two circles of radius 1 (see text book). Clearly, the sum of these diameters is an infinite series which converges to 1. We have to prove that $d_n = \frac{1}{n(n+1)}$ for all n. We'll prove that $d_1 = 1/2$ (you should verify that $d_2 = 1/6$ yourselves), and then show that the pattern of d_n's continues: that is, given that d_1 upto d_n are precisely the first n terms of our series above, then the next diameter d_{n+1} will be the $(n+1)$-st term.

Start the pattern: Draw a right angled triangle with vertices at the center of one of the big circles, at the point of tangency of the big circles, and at the center of the first of the packed circles. The height of this triangle is $1 - \frac{d_1}{2}$, its length is 1, and its hypothenuse is $1 + \frac{d_1}{2}$. Pythagoras theorem gives us

$$\left(1 - \frac{d_1}{2}\right)^2 + 1^2 = \left(1 + \frac{d_1}{2}\right)^2$$

and we can solve this to get $d_1 = 1/2$.



The general pattern: Suppose that we have already proven that the first n diameters are equal to the first n terms of the telescoping series above. This means that the combined height of the first n packed circles is just $s_n = 1 - 1/(n+1)$. Thus if we construct a right triangle as before, but now use the center of the $(n+1)$-st packed circle as one vertex (see picture), we get a height of $1/(n+1) - d_{n+1}/2$, a length of 1, and an hypothenuse of $1 + d_{n+1}/2$. Again Pythagoras theorem gives us

$$\left(\frac{1}{n+1} - \frac{d_{n+1}}{2}\right)^2 + 1^2 = \left(1 + \frac{d_{n+1}}{2}\right)^2$$

which can be solved for d_{n+1} to get $d_{n+1} = \frac{1}{(n+1)(n+2)}$. **Do the algebra!** We're done!